Semantical study of intuitionistic modal logics

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January 16, 2012

Abstract

We investigate several intuitionistic modal logics (IMLs), mainly from semantical viewpoint. In these decades, in the area of type theory for programming language, various type systems corresponding (via Curry-Howard correspondence) to IMLs have been considered. However, IMLs arising from such a context are not extensively studied before. We try to understand the logical meaning of such IMLs, and what kind of structure are behind IMLs motivated from computation.

The technical materials appearing in the thesis are divided into three parts: (1) proof systems and Kripke semantics for an intuitionistic version of linear-time temporal logic (LTL), which corresponds to a lambda-calculus for binding-time analysis; (2) correspondence theory in a certain Kripke semantics for IML; and (3) a new representation of existing Kripke semantics for IML by using neighborhood semantics (also called Scott-Montague semantics), and investigations of the relationship between them.

Through these technical developments, the following observation arises: there are actually two possible meanings of the assertion "a formula A is true at a possible world x." This explains the difference between the traditional modal logic and IMLs considered in this thesis, and several difficulties that emerge in establishing meta-theoretical results such as completeness and cut-elimination.

In the thesis we also show that the existing Kripke model can be transformed into an equivalent neighborhood model, and each neighborhood model satisfying a certain condition can be transformed into an equivalent Kripke model. This mutual translation clearly explains how "non-standard" behavior of modalities (captured well by neighborhood semantics) has been simulated in the existing Kripke semantics.

Contents

Chapter 1

Introduction

1.1 Background and Motivation

Recently, intuitionistic modal logics (IMLs, for short) have been used in several areas of theoretical computer science. For example, in the context of typed lambda calculi, connections between IMLs and binding-time analysis [13], staged computation [14, 45, 40], computational effects [26, 21, 7], distributed computation [27], and security [3, 36] have been studied. In the area of formal verification, an application to hardware verification [16] and reasoning about truncated execution paths [24] have been considered. Also, recently it has been suggested that IML can be a better language than classical one to describe security policy [2, 9].

The ultimate goal of this thesis is to provide a clear understanding of these IMLs from a logical point of view. Current state of the study of IML is fairly unclear. There are several different versions of IML, and many variants of Kripke semantics are considered. Although some fundamental results such as completeness, decidability, and finite model property are established for each of them, there are no unifying framework for IML, nor are there common agreement on what an IML should be, and how it is related to classical modal logics.

In particular, this work stresses the semantic aspect of the existing IMLs. Compared to the syntactic approach, it seems that the semantics of IMLs is not very seriously studied. Although some previous researches consider semantics as well as proof systems, it is still unclear what is the meaning of those semantics. This is very comparative to the case of classical modal logic, to which Kripke semantics provides a quite clear, intuitive way of viewing the logic in consideration.

1.2 Overview of the Thesis

The rest of the thesis is organized as follows.

Chapter 2 provides preliminaries for later chapters. We review existing approaches to possible world semantics for IML, and introduce the relational semantics (and its variants) we consider in the thesis. Complete axiomatizations for relational semantics and its variants are given in Hilbert-style. Our logic does not satisfy the principle $\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$, which we call the *distributivity law*. This is usually accepted in classical modal logic, but intuitionistically it is not always the case.

In Chapter 3, we introduce an intuitionistic version of LTL with the "next" temporal operator. This logic has first appeared in the application to binding-time analysis [13]. We define Kripke semantics for it, and prove soundness and completeness of a certain Hilbert-style proof system. We also consider natural deduction and sequent calculus, and prove the cut-elimination theorem for the sequent calculus.

In Chapter 4, we consider a problem concerning so-called correspondence theory. In classical modal logic, it is well-known that some properties of Kripke frames are characterized by modal axioms; for example, reflexivity and transitivity are characterized by $p \to \Diamond p$ (axiom **T**) and $\Diamond\Diamond p \to \Diamond p$ (axiom **4**), respectively. This relationship between properties of frames and modal axioms is called correspondence, and is already studied well in the classical setting. However, in the intuitionistic setting the same relationship does not hold in general. For example, reflexivity is no longer characterized by $p \to \Diamond p$

(precisely speaking, this characterization may be true, depending on the choice of the definition of semantics, but it is not the case for our semantics defined in Chapter 2).

We will consider the following problem: to what extent are the classical correspondence results valid in an intuitionistic setting? We define a certain class of modal axioms, and show that any axiom belonging to that class has the same classical and intuitionistic correspondents.

In Chapter 5 we define neighborhood semantics for IML. Neighborhood semantics has originally been considered in classical setting as a semantics of non-normal modal logics (modal logics which Kripke semantics cannot handle) [12, 29], and ours is its intuitionistic analogue.

After defining neighborhood semantics, we give a complete axiomatization of this semantics, and discuss the relationship between neighborhood and relational semantics. Interestingly, it turns out that these two semantics are almost equivalent in our settings.

More precisely, "almost equivalent" means:

1. given a relational model, we can construct an equivalent neighborhood model, and
2. given a neighborhood model satisfying a certain condition (every world has at least one neighborhood), we can transform it into an equivalent relational model.

This is a surprising result because classically there is a big difference between Kripke and neighborhood semantics.

We also consider the logic and semantics in classical setting. Interestingly, the classical version of our logic is not a normal modal logic, even in the case of relational semantics! Instead, we obtain a modal logic with both normal and non-normal modalities, together with additional axiom that specifies how these modalities interact. It should also be noted that, if we make the semantics classical, the corresponding modification in the syntactic side can be done by just allowing the classical reasoning (the principle of excluded middle, for example).

In Chapter 6, we discuss the interpretations of the technical results described in the preceding chapters. We introduce the notions of internal and external observers, and give an intuitively understandable interpretation of the $\Diamond$ modality without the distributivity law having appeared in the preceding chapters.

One important observation here is that the failure of the distributivity law in IML is caused by the difference of the meaning of modalities or the notion of "truth at a possible world," rather than the fact that the base logic is intuitionistic. This is suggested by the result in Chapter 5, where it is proved that it is possible to move from intuitionistic to classical setting without losing allowing the distributivity law. If the behavior were a consequence of intuitionistic base logic, such a behavior would disappear if we make the base logic classical.

Another question naturally arises from this argument. What is the intuitive meaning of such a phenomenon? What is the difference between the "usual" modalities (that can be well understood in terms of Kripke semantics) and "unusual" ones? Is it possible to give a good account of considering possible world semantics for such "unusual" modalities, or is it just a technical apparatus that is hard to explain an intuitive meaning? We consider the notion of "observers" to answer these questions.

We propose a distinction between *internal* and *external* observers. This sheds light on the subtlety of the meaning of modalities in IML that have not been made clear yet. In short, an external observer observes possible worlds from outside the system, whereas an internal observer sits in a particular world, and cannot move to other worlds. In the traditional Kripke semantics the notion of observer is left implicit, but we can regard it as a semantics based on an implicit external observer's viewpoint. It is the internal observer's viewpoint that brings the unusual behavior to $\Diamond$ modality.

To summarize, in addition to the technical results, this work makes a new finding that some of the existing IMLs have modalities which have different meaning than the traditional modal logics. This point of view brings a new way of viewing IMLs and advances our understanding of IMLs.

Chapter 2

Kripke Semantics for Intuitionistic Modal Logics

2.1 Introduction

2.1.1 Overview

The main objective in this chapter is to introduce a Kripke-style semantics of IML that will be used in later chapters. The most part of this chapter is a rearrangement of the existing literature.

We introduce two IMLs in this chapter. One of them is axiomatized by a fairly small set of axioms, and is called IK^-. The other is obtained by adding $\neg\Diamond\bot$ to IK^-, and denoted by $\mathrm{IK}^- + \mathbf{N}_\Diamond$.

Kripke semantics for $\mathrm{IK}^- + \mathbf{N}_\Diamond$ is already known from Wijesekera's work [42], and the result presented in this chapter is the same as his. However, we give a different proof of completeness, being based on the idea by Alechina et al. [4] and Hilken [20].

The semantics is based on a Kripke structure with two accessibility relations, R and $\leq$. The point is that both $\leq$ and R are used to interpret $\Diamond$ modality: $\Diamond A$ is true at x if and only if for all $x' \geq x$ there exists y such that $x'\ R\ y$ and A is true at y. Although technically this choice seems successful, there is a drawback: the usual connection between $\Diamond$ and $\exists$ is lost, and it is not clear what is the meaning of $\Diamond$ with such an interpretation (we will discuss this question in Chapter 6).

Kripke semantics defined in this way can falsify the distributivity law as expected, but it validates $\neg\Diamond\bot$. This formula is not always acceptable under the "formulas-as-types" interpretation. If we regard $\Diamond A$ as a type of code fragments that can be executed under a certain constraint, then $\neg\Diamond\bot$ is not necessarily plausible (for more details, see Section 3.1, where we discuss a similar matter concerning the "next" operator in LTL).

To fix the semantics so that it does not validate $\neg\Diamond\bot$, we would need some trick. Following the previous approach [4, 16], we consider "fallible worlds," where all formulas (even $\bot$) become true. By imposing such worlds, we can construct a model where $\Diamond\bot$ can be true, therefore $\neg\Diamond\bot$ is falsified. After this modification we can show that IK^- is sound and complete.

2.1.2 Organization of the Chapter

In Section 2.2 we review previous approaches. In Section 2.3 we introduce Hilbert style proof systems for several IMLs. In Section 2.4 we define Kripke semantics for IMLs. We introduce two types of semantics, fallible and non-fallible ones. In Section 2.5 we prove completeness theorem for each of the two types of semantics. Finally in Section 2.6 we briefly summarize the chapter and make a few remarks.

2.2 Existing Intuitionistic Modal Logics

2.2.1 Preliminary Definitions

Throughout the rest of the thesis we use the following modal languages.

- We use PV for a fixed infinite set of propositional variables. Propositional variables are ranged over by p, q.
- $\mathbf{L}(O_1, \ldots, O_n)$ is the set of formulas built from elements of PV, $\top$ and $\bot$ with logical connectives $\wedge, \vee, \rightarrow$ and unary modalities $O_1, \ldots, O_n$. More precisely, $\mathbf{L}(O_1, \ldots, O_n)$ is the language defined by the following syntax:

$$A, B ::= \mathrm{PV} \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid O_i A,$$

  where $1 \leq i \leq n$.
- $\neg A$ is an abbreviation of $A \rightarrow \bot$.

Notation. We denote the composition of binary relations by $(\cdot\,;\cdot)$. That is, if R and S are binary relations on a set X, then $x\ (R\,;S)\ y \iff \exists z.(x\ R\ z\ S\ y)$. We sometimes apply this notation when one of R or S is a function, regarding it as a special case of a binary relation.

Notation. For a binary relation R on a set X and an element $x \in X$, its image $\{y \mid x\ R\ y\}$ is denoted by $R[x]$.

2.2.2 Proof Systems

Here we will consider the usual modal language $\mathbf{L}(\Box, \Diamond)$. Note that $\Box$ and $\Diamond$ are both introduced as primitive operators. Classically, if we have one of them we can define the other operator by $\Box = \neg\Diamond\neg$ and $\Diamond = \neg\Box\neg$. However they are analogues of de-Morgan's laws, so it does not seem reasonable to impose such principles in the intuitionistic setting.

To formalize IML syntactically, it seems relatively common to use Hilbert-style axiomatization. We begin with the axiomatization of classical modal logic K, typically given by the following axioms and rules (assuming $\Diamond$ is defined as $\neg\Box\neg$):

- all substitution instances of classical tautologies;
- axiom **K**: $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$;
- modus ponens: if $A \rightarrow B$ and A, then B;
- necessitation: if A, then $\Box A$.

This gives the weakest normal modal logic K, and by adding extra axioms we can obtain various well-known modal logics.

When a system is given in this way, the simplest way to make it intuitionistic is to replace "classical tautologies" with "intuitionistic tautologies," and leave other clauses unchanged. This approach works to some extent, but is not satisfactory. The problem is that these axioms and rules do not mention $\Diamond$. It seems work well when we work with $\Box$-fragment, but if $\Diamond$ is involved, we need more axioms to obtain an appropriate axiomatization. If we introduce $\Diamond$ and $\Box$ independently, axiom **K** proves nothing about $\Diamond$.

To fix this problem, several additional axioms concerning $\Diamond$ are considered. Axiom $\Box(p \rightarrow q) \rightarrow \Diamond p \rightarrow \Diamond q$ is a quite common axiom, and it is included in most of the existing Hilbert-style proof systems. $\neg\Diamond\bot$ is another axiom which is often introduced as an axiom, but some of the existing work rejects this principle [4].

There are two other axioms which seem to be considered in the literature: $\Diamond(p \vee q) \rightarrow \Diamond p \vee \Diamond q$ and $(\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$. Although these two axioms can be seen as modal versions of the intuitionistic theorems $(\exists x.\varphi \vee \psi) \rightarrow (\exists x.\varphi) \vee (\exists x.\psi)$ and $((\exists x.\varphi) \rightarrow (\forall x.\psi)) \rightarrow \forall x.(\varphi \rightarrow \psi)$, they are occasionally rejected. In particular, the reason why $\Diamond(p\vee q) \rightarrow \Diamond p \vee \Diamond q$ is implausible from a type-theoretic viewpoint is quite similar to the point discussed in Chapter 3.

Although much work on IMLs is done in Hilbert-style, other styles of formalization are also considered. Natural deduction system is studied in relation with the computational interpretation of intuitionistic logic. For example, Martini and Masini [25] considered natural deduction style formalization of several IMLs and discussed its proof normalization and computational interpretation by giving the corresponding term calculi. Sequent calculus is also studied by Wijesekera [42]. Also, Pfenning and Davies considered judgmental formalization of intuitionistic S4 [31].

2.2.3 Kripke Semantics

There have been several Kripke-style approaches to IML. In the literature we can find two types of approaches:

1. Interpret IML in a model of intuitionistic first-order logic, by correlating modal operators and first-order quantifiers.
2. Start from Kripke semantics in the classical sense, and augment it with some structure which represents intuitionistic counterpart.

The basic idea behind the first approach is the classical theory which relates classical modal logic and classical first-order logic. This theory shows that $\Box$ and $\Diamond$ can be regarded as $\forall$ and $\exists$, respectively. Kripke semantics based on this idea is studied by Ewald [15] and Simpson [37], for example.

In the second approach, there are some possible choices on an additional structure to be added. Perhaps the most common choice is to add another accessibility relation $\leq$ to the Kripke frame. This additional relation is taken from the standard Kripke semantics for intuitionistic logic. As a result, Kripke frames given in this way have two accessibility relations, and are sometimes called birelational frames.

Instead of adding an intuitionistic accessibility relation, changing the set of truth-values is another way previously considered to introduce intuitionistic behavior. Namely, we can take truth-values from some Heyting-algebra. This approach is studied by Ono as "modal-type Kripke models" [28].

Variants of birelational frames Below we will consider a triple $\langle W, \leq, R\rangle$.[1] Here $\leq$ and R are taken from the Kripke semantics for intuitionistic logic and modal logic, respectively.

A problem in introducing modalities in this way is that the classical truth conditions for modalities

$$x \Vdash \Box A \iff \forall y.\ (x\ R\ y \implies y \Vdash A),$$
$$x \Vdash \Diamond A \iff \exists y.\ (x\ R\ y \text{ and } y \Vdash A)$$

breaks the heredity condition

$$x \leq y \text{ and } x \Vdash A \implies y \Vdash A,$$

which is expected in the Kripke semantics for intuitionistic logic.

To remedy this problem, there are several choices in two points: how the modal operators should be interpreted, and how the intuitionistic accessibility $\leq$ should interact with the modal one R. Roughly speaking, there are three combinations previously considered:

1. There are no constraint on how $\leq$ and R are related, and both $\Box$ and $\Diamond$ are interpreted "intuitionistically," that is, by conditions of the form "for all x' with $x \leq x'$ it holds that…"
2. $\leq$ and R need to satisfy two inclusions $(\geq ; R) \subseteq (R ; \geq)$ and $(\leq ; R) \subseteq (R ; \leq)$, and only $\Box$ is interpreted intuitionistically.
3. Two equalities $(\leq ; R ; \leq) = R$ and $(\geq ; R ; \geq) = R$ are required, and both $\Box$ and $\Diamond$ are interpreted classically. In this approach, sometimes distinct two accessibility relations $R_\Box$ and $R_\Diamond$ are used to interpret $\Box$ and $\Diamond$, respectively. In such a case, the constraints become $(\leq ; R_\Box ; \leq) = R_\Box$ and $(\geq ; R_\Diamond ; \geq) = R_\Diamond$.

For example, Wijesekera [42] takes the first approach. The semantics for intuitionistic S4 by Alechina et al. [4] can also be classified in this category; they needed the condition $(R ; \leq) \subseteq (\leq ; R)$, but this seems to be introduced for the soundness of the axiom **4**.

The second one is taken by Plotkin and Stirling [32] and Simpson [37]. It seems that, although their representation of semantics is birelational, they are motivated by the idea of using first-order logic mentioned above. The inclusions $(\geq ; R) \subseteq (R ; \geq)$ and $(\leq ; R) \subseteq (R ; \leq)$ are its natural consequences.

The third approach can be seen in work by Sotirov [38], Božič and Došen [10], and Wolter and Zakharyaschev [44, 43]. "Intuitionistic-type Kripke models" studied by Ono [28] would also be of this category, although at first sight his representation does not seem so.

A more extensive reviews of the previous approaches to IMLs before early '90s can be found in Simpson's thesis [37].

[1] Another approach often seen in the literature is to consider first-order Kripke structure, and interpret $\Box$ and $\Diamond$ as $\forall$ and $\exists$ in it. This approach has been considered by Ewald [15] and Simpson [37], for example.

$(\mathbf{N}_{\Diamond\Box})$	$\Diamond\bot \to \Box\bot$	$(\mathbf{K})$	$\Box(p \to q) \to \Box p \to \Box q$
$(\mathbf{N}_{\Diamond})$	$\neg\Diamond\bot$	$(\mathbf{K}_{\Diamond})$	$\Box(p \to q) \to \Diamond p \to \Diamond q$
$(\mathbf{PEM})$	$p \vee \neg p$	$(\mathbf{Dual})$	$\Diamond p \leftrightarrow \neg\Box\neg p$

Figure 2.1: Some axioms of our interest

2.3 Proof Systems

Definition 2.3.1.

1. A set $\Lambda \subseteq \mathbf{L}(\Box, \Diamond)$ is said to be an $\mathbf{L}(\Box, \Diamond)$-*logic* if it contains all intuitionistic tautologies, and closed under the following rules:
   (a) uniform substitution: if $A \in \Lambda$, then $A[p := B] \in \Lambda$, where $A[p := B]$ denotes the formula obtained by replacing all occurrences of p in A with B;
   (b) modus ponens: if $A \in \Lambda$ and $A \to B \in \Lambda$, then $B \in \Lambda$;
   (c) necessitation: if $A \in \Lambda$, then $\Box A \in \Lambda$.
2. Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic. A set $\Lambda' \subseteq \mathbf{L}(\Box, \Diamond)$ is said to be a Λ-*logic* if Λ' is an $\mathbf{L}(\Box, \Diamond)$-logic and $\Lambda \subseteq \Lambda'$.
3. Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic. A set $\Gamma \subseteq \mathbf{L}(\Box, \Diamond)$ is said to be a Λ-*theory* if it contains Λ and closed under modus ponens.

In Figure 2.1, we list axioms that will appear later.

Definition 2.3.2. IK^- is the smallest $\mathbf{L}(\Box, \Diamond)$-logic containing axioms $\mathbf{K}$ and $\mathbf{K}_{\Diamond}$.

Definition 2.3.3. Let A be a formula, and Λ be an IK^--logic. We define $\Lambda + A$ as the smallest Λ-logic containing A.

Notation.

1. Let Γ be a set of formulas, and A a formula. We write
$$\Gamma \vdash_\Lambda A$$
when A is contained in any Λ-theory that contains Γ (in other words, A can be proved in Λ from Γ).
2. We use the common convention on the left-hand side of $\vdash$: we write $\Gamma, A \vdash B$ for $\Gamma \cup \{A\} \vdash B$.
3. We omit the subscript of $\vdash_\Lambda$ if Λ is clear from the context.

Theorem 2.3.4 (Deduction theorem)**.** *Let Λ be a $\mathbf{L}(\Box, \Diamond)$-logic.*

1. *Let Γ be a Λ-theory. Then, $A \to B \in \Gamma$ if and only if all Λ-theories Δ containing Γ and A contain B.*
2. *Let Γ be a set of formulas, and A and B arbitrary formulas. Then*
$$\Gamma \vdash_\Lambda A \to B \iff \Gamma, A \vdash_\Lambda B.$$

Proof. Since the second part is an easy consequence of the first part, we only show the first. To prove "if" part, let $\Delta = \{C \mid A \to C \in \Gamma\}$. Then Δ is a theory containing Γ and A. Then, from assumption, Δ contains B, and this means $A \to B \in \Gamma$ by definition of Δ. For the "only if" part, use the fact that any theory is closed under modus ponens. $\Box$

Theorem 2.3.5 (Compactness)**.** *Let Λ be a $\mathbf{L}(\Box, \Diamond)$-logic, Γ a set of formulas, and A a formula such that $\Gamma \vdash_\Lambda A$. Then there exists a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_\Lambda A$.*

Proof. Consider $\Delta := \{A \mid \Gamma' \vdash_\Lambda A$ for some finite $\Gamma' \subseteq \Gamma\}$. Then, Δ is a theory containing Γ. Therefore $\Gamma \vdash_\Lambda A$ implies $A \in \Delta$, from which the consequence follows. $\Box$

2.4 Definition of Kripke Semantics

2.4.1 Non-Fallible Kripke Semantics

Definition 2.4.1. An *intuitionistic relational frame* (IR-frame, for short) is a triple $\langle W, \leq, R \rangle$ of a non-empty set W, a preorder $\leq$ on W, and a binary relation R on W.

Definition 2.4.2. 1. For an IR-frame $\mathcal{R} = \langle W, \leq, R \rangle$, an $\mathcal{R}$*-valuation* is a map V from PV to $\mathcal{P}(W)$.

2. An $\mathcal{R}$-valuation V is said to be *admissible* if $V(p)$ is upward-closed for all propositional variables p.

Definition 2.4.3. An *intuitionistic relational model* (IR-model) is a pair $\langle \mathcal{R}, V \rangle$ of an IR-frame $\mathcal{R}$ and an admissible $\mathcal{R}$-valuation V.

Definition 2.4.4. Let $\langle \mathcal{R}, V \rangle$ be an IR-model. We can define the satisfaction relation, denoted by $\Vdash_{\mathrm{r}}$, as follows:

$$\begin{aligned}
\mathcal{R}, V, x &\nVdash_{\mathrm{r}} \bot; \\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} p &\iff x \in V(p); \\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} A \wedge B &\iff \mathcal{R}, V, x \Vdash_{\mathrm{r}} A \text{ and } \mathcal{R}, V, x \Vdash_{\mathrm{r}} B; \\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} A \vee B &\iff \mathcal{R}, V, x \Vdash_{\mathrm{r}} A \text{ or } \mathcal{R}, V, x \Vdash_{\mathrm{r}} B; \\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} A \to B &\iff \forall y \geq x.(\mathcal{R}, V, y \Vdash_{\mathrm{r}} A \implies \mathcal{R}, V, y \Vdash_{\mathrm{r}} B); \\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} \Box A &\iff \forall y \geq x. \forall z.(y \; R \; z \implies \mathcal{R}, V, z \Vdash_{\mathrm{r}} A); \\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} \Diamond A &\iff \forall y \geq x. \exists z.(y \; R \; z \text{ and } \mathcal{R}, V, z \Vdash_{\mathrm{r}} A).
\end{aligned}$$

Below we sometimes suppress $\mathcal{R}$ and V if they are clear from the context.

Here are some remarks on the interpretation of $\Diamond$.

Remark 2.4.5. 1. Intuitionistic interpretation of $\Diamond$ requires to consider all w' above w. This comes from heredity condition usually assumed in intuitionistic logics.

2. The interpretation of $\Diamond$ above is the same as the one previously considered to reject distributivity law $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$. This formula is a theorem in classical modal logic, but sometimes it is considered intuitionistically unreasonable [42, 4].

It is easy to verify that the heredity condition

$$\mathcal{R}, V, x \Vdash_{\mathrm{r}} A \text{ and } x \leq y \implies \mathcal{R}, V, y \Vdash_{\mathrm{r}} A$$

holds for all formulas $A \in \mathbf{L}(\Box, \Diamond)$.

Definition 2.4.6. A formula A is said to be *valid in* $\mathcal{R}$ if $\mathcal{R}, V, x \Vdash_{\mathrm{r}} A$ for any $\mathcal{R}$-valuation V and a world x of $\mathcal{R}$.

Definition 2.4.7. Let A be a formula in $\mathbf{L}(\Box, \Diamond)$.

1. Let $\langle \mathcal{R}, V \rangle$ be an IR-model. A is said to be *true in* $\langle \mathcal{R}, V \rangle$ if $\mathcal{R}, V, x \Vdash_{\mathrm{r}} A$ for all $x \in W$.

2. Let $\mathcal{K}$ be a class of IR-models. A is said to be *true in* $\mathcal{K}$ if and only if it is true in all models of $\mathcal{K}$.

2.4.2 Fallible Kripke Semantics

It is easily checked that the non-fallible semantics validates the axiom $\mathbf{N}_{\Diamond}$. To avoid this, we consider "fallible" Kripke semantics. Our version is variant of Fairtlough and Mendler's semantics for lax logic [16] and Alechina et al.'s semantics for constructive S4 [4].

Definition 2.4.8. A *fallible intuitionistic relational frame* (fallible IR-frame, for short) is a quadruple $\langle W, \leq, R, F \rangle$ of a non-empty set W, a preorder $\leq$ on W, a binary relation R on W, and a subset F of W. Additionally, we require the following conditions:

- F is closed under $\leq$: if $x \in F$ and $x \leq y$, then $y \in F$.
- F is closed under R: if $x \in F$ and $x \mathrel{R} y$, then $y \in F$.
- R is serial on F: if $x \in F$, then there exists $y \in F$ such that $x \mathrel{R} y$.

Definition 2.4.9. 1. For a fallible IR-frame $\mathcal{R} = \langle W, \leq, R, F\rangle$, an $\mathcal{R}$*-valuation* is a map V from PV to $\mathcal{P}(W)$ such that $F \subseteq V(p)$ for all $p \in \mathrm{PV}$.

2. An $\mathcal{R}$-valuation V is said to be *admissible* if $V(p)$ is upward-closed for all propositional variables p.

Definition 2.4.10. An *fallible intuitionistic relational model* (fallible IR-model) is a pair $\langle \mathcal{R}, V\rangle$ of a fallible IR-frame $\mathcal{R}$ and an admissible $\mathcal{R}$-valuation V.

Definition 2.4.11. Let $\langle \mathcal{R}, V\rangle$ be a fallible IR-model. We can define the satisfaction relation in the same way as the non-fallible case, except for the case of $\bot$:

$$\mathcal{R}, V, x \Vdash_{\mathrm{r}} \bot \iff x \in F.$$

The notions of validity and truth are defined in the same way as Definition 2.4.6 and Definition 2.4.7. By using the fact that $\bot$ is true in a fallible world, it is easy to verify that $\neg\Diamond\bot$ is not necessarily true in a fallible IR-model.

2.4.3 IM-Frames

In later chapters, we will encounter a more restricted class of IR-frames, which we will call IM-frames. Originally, IM-frames have been defined by Wolter and Zakharyaschev, and used to discuss the relationship between a certain IML and classical bimodal logic [44]. They originally studied not only Kripke frames, but also general frames, and they defined IM-frames as general frames equipped with an order. However, by an abuse of terminology, we simply say IM-frame to mean Kripke IM-frame in their terminology. Since we do not consider general frames here, no confusion occurs.

Definition 2.4.12. 1. An *IM-frame* is a (non-fallibe) IR-frame $\langle W, \leq, R\rangle$ that satisfies $(\leq;R;\leq) = R$.

2. An *IM-model* is a (non-fallibe) IR-model $\langle \mathcal{R}, V\rangle$ where $\mathcal{R}$ is an IM-frame.

Since an IM-model is an IR-model satisfying a particular condition, the notions of satisfaction relation, validity and truth are naturally derived from Definitions 2.4.4, 2.4.6, and 2.4.7.

Although the same clauses of truth conditions are applicable to IM-models, we can actually say that the interpretation of $\Box$ can be simplified in an IM-model. That is, to check that $x \Vdash_{\mathrm{r}} \Box A$, we need to consider R-successors of x only. More formally, the following holds.

Proposition 2.4.13. *If $\mathcal{R}$ is an IM-model, then*

$$\mathcal{R}, V, x \Vdash_{\mathrm{r}} \Box A \iff \forall z.(x \mathrel{R} z \implies \mathcal{R}, V, z \Vdash_{\mathrm{r}} A).$$

Proof. Left to right is clear. For the converse, note that if $x \leq y \mathrel{R} z$ then

$$(x, z) \in (\leq ; R) \subseteq (\leq ; R ; \leq) = R,$$

therefore $x \mathrel{R} z$. The last equality comes from the definition of IM-frame. $\square$

2.5 Completeness

In this section we are going to prove the following theorem.

Theorem 2.5.1. *For any formula A, the following are equivalent:*

1. *A is a theorem of* IK^-*;*
2. *A is valid in every fallible IR-frame.*

It is easy to check that 2 follows from 1. For the converse, we construct a canonical model.

2.5.1 Preliminary Definitions and Lemmas

Here we consider arbitrary $\mathbf{L}(\Box, \Diamond)$-logic Λ, rather than only IK^-. Although we use only the case $\Lambda = \mathrm{IK}^-$, the results in this subsection will be used later.

Definition 2.5.2. Let Γ and Θ be sets of formulas. Then we define

$$\Gamma \vdash_\Lambda \Theta \iff \exists B_1, \ldots, B_m \in \Theta, (\Gamma \vdash_\Lambda B_1 \vee \cdots \vee B_m).$$

Here we require $m > 0$, that is, it is never the case that $\Gamma \vdash \emptyset$ even if Γ is inconsistent. The convention like $A \vdash \Theta$ is understood as usual.

Remark 2.5.3. This definition is consistent with the previous one in the sense that

$$\Gamma \vdash \{A\} \text{ in the new definition } \iff \Gamma \vdash A \text{ in the previous definition.}$$

Definition 2.5.4. A pair (Γ, Θ) of sets of formulas is said to be Λ-*consistent* if $\Gamma \nvdash_\Lambda \Theta$.

Definition 2.5.5. Let Λ be a $\mathbf{L}(\Box, \Diamond)$-logic and Γ a Λ-theory.

1. Γ is said to be *consistent* if it does not contain $\bot$. Otherwise Γ is said to be *inconsistent*.
2. Γ is said to be *prime* if it satisfies the following property: if $A \vee B \in \Gamma$, then either $A \in \Gamma$ or $B \in \Gamma$. Here Γ may be inconsistent.
3. A (possibly empty) set of formulas Θ is said to be a Λ-*co-theory* if it satisfies:

$$\forall A, (A \vdash_\Lambda \Theta \implies A \in \Theta).$$

Definition 2.5.6. Let Γ be a set of formulas. Then we define

$$\Box^{-1}\Gamma := \{A \mid \Box A \in \Gamma\};$$
$$\Diamond^{-1}\Gamma := \{A \mid \Diamond A \in \Gamma\}.$$

Lemma 2.5.7. *Let Λ be a $\mathbf{L}(\Box, \Diamond)$-logic containing axiom $\mathbf{K}$, and Γ a Λ-theory. Then, $\Box^{-1}\Gamma$ is also a Λ-theory.*

Proof. Since Λ is closed under $\Box$ and $\Lambda \subseteq \Gamma$, it is clear that $\Lambda \subseteq \Box^{-1}\Gamma$. So it suffices to show that $\Box^{-1}\Gamma$ is closed under modus ponens. Suppose $A \to B \in \Box^{-1}\Gamma$ and $A \in \Box^{-1}\Gamma$. Then $\Box(A \to B) \in \Gamma$ and $\Box A \in \Gamma$. Since Λ contains K, it follows that $\Box B \in \Gamma$, hence $B \in \Box^{-1}\Gamma$. So $\Box^{-1}\Gamma$ is closed under modus ponens, as required. $\Box$

Lemma 2.5.8 (Extension lemma)**.** *Let Λ be a $\mathbf{L}(\Box, \Diamond)$-logic, and (Γ, Θ) an arbitrary Λ-consistent pair. Then there exists a prime Λ-theory $\Gamma' \supseteq \Gamma$ and a Λ-co-theory $\Theta' \supseteq \Theta$ such that (Γ', Θ') is also Λ-consistent.*

Proof. Let Γ_0 be the least Λ-theory containing Γ, and Θ_0 the least Λ-co-theory containing Θ, that is,

$$\Gamma_0 = \{A \mid \Gamma \vdash_\Lambda A\};$$
$$\Theta_0 = \{A \mid A \vdash_\Lambda \Theta\}.$$

We first prove that (Γ_0, Θ_0) is Λ-consistent. Otherwise, there exists some $B_1, \ldots, B_n \in \Gamma$ and $C_1, \ldots, C_m \in \Theta$ such that

$$\Gamma \vdash_\Lambda B_1, \ldots, \Gamma \vdash_\Lambda B_n,$$
$$C_1 \vdash_\Lambda \Theta, \ldots, C_m \vdash_\Lambda \Theta,$$
$$B_1, \ldots, B_n \vdash_\Lambda C_1 \vee \cdots \vee C_m.$$

This implies $\Gamma \vdash_\Lambda \Theta$, a contradiction.

Next, consider the set

$$\{\Delta \supseteq \Gamma_0 \mid \Delta \text{ is a theory and } (\Delta, \Theta_0) \text{ is } \Lambda\text{-consistent}\}.$$

Then this is non-empty and inductive under set-inclusion. Let Γ' be its maximal element. Then it is a routine to check that Γ' is a prime theory. $\Box$

Corollary 2.5.9. *Let Λ be a $\mathbf{L}(\Box, \Diamond)$-logic, and Γ an arbitrary set of formulas such that $A \notin \Gamma$. Then there exists a prime Λ-theory $\Gamma' \supseteq \Gamma$ such that $A \notin \Gamma'$.*

Proof. Let $\Theta = \{A\}$ in the previous lemma. $\square$

2.5.2 Canonical Model Construction

Definition 2.5.10 (Canonical Frame for IK^-)**.** We define the canonical frame $\langle W, \leq, R, F\rangle$ as follows:

- W is the set of pairs (Γ, Θ), where
  - Γ is a prime theory,
  - Θ is a co-theory, and
  - $(\Gamma, \Diamond\Theta)$ is consistent.
- $(\Gamma, \Theta) \leq (\Gamma', \Theta')$ if and only if $\Gamma \subseteq \Gamma'$.
- $(\Gamma, \Theta)\ R\ (\Gamma', \Theta')$ if and only if $\Box^{-1}\Gamma \subseteq \Gamma'$ and (Γ', Θ) is consistent.
- $F = \{(\nabla, \emptyset)\}$, where ∇ is the set of all formulas.

Lemma 2.5.11. *The tuple $\langle W, \leq, R, F\rangle$ defined above is a fallible IR-frame.*

Lemma 2.5.12. *Let $\mathcal{R} = \langle W, \leq, R, F\rangle$ be the canonical frame defined above, and V be the canonical valuation given by*

$$V(p) = \{(\Gamma, \Theta) \mid p \in \Gamma\}.$$

Then it holds that

$$\mathcal{R}, V, (\Gamma, \Theta) \Vdash_{\mathrm{r}} A \iff A \in \Gamma$$

for every formula A.

Proof. We proceed by induction on A. The cases when A is an atomic formula, $\bot$, conjunction, or disjunction are trivial.

$\underline{A \to B \in \Gamma \implies (\Gamma, \Theta) \Vdash_{\mathrm{r}} A \to B}$:

$$\begin{aligned} &(\Gamma, \Theta) \leq (\Gamma', \Theta') \Vdash_{\mathrm{r}} A \\ \implies &A, A \to B \in \Gamma' \\ \implies &B \in \Gamma'. \end{aligned}$$

So $(\Gamma, \Theta) \Vdash_{\mathrm{r}} A \to B$.

$\underline{A \to B \notin \Gamma \implies (\Gamma, \Theta) \nVdash_{\mathrm{r}} A \to B}$: If $A \to B \notin \Gamma$, then $\Gamma, A \nvdash B$. So applying extension lemma to the consistent pair $(\Gamma \cup \{A\}, \{B\})$ we obtain a consistent pair (Γ', Θ') where Γ' is a prime theory. Since this Γ' satisfies $\Gamma \subseteq \Gamma'$, $A \in \Gamma'$, and $B \notin \Gamma'$, we have

$$(\Gamma, \Theta) \leq (\Gamma', \Theta') \nVdash_{\mathrm{r}} A \to B.$$

$\underline{\Box A \in \Gamma \implies (\Gamma, \Theta) \Vdash_{\mathrm{r}} \Box A}$: Suppose $(\Gamma, \Theta) \leq (\Gamma'', \Theta'')\ R\ (\Gamma', \Theta')$. Then we have $\Box^{-1}\Gamma \subseteq \Gamma'$, and

$$\Box^{-1}\Gamma \subseteq \Gamma' \implies A \in \Gamma' \implies (\Gamma', \Theta') \Vdash_{\mathrm{r}} A.$$

$\underline{\Box A \notin \Gamma \implies (\Gamma, \Theta) \nVdash_{\mathrm{r}} \Box A}$: If $\Box A \notin \Gamma$, then $A \notin \Box^{-1}\Gamma$. From extension lemma there exists some prime theory $\Delta \supseteq \Box^{-1}\Gamma$ with $A \notin \Delta$. For such Δ, we have

$$(\Gamma, \Theta) \leq (\Gamma, \emptyset)\ R\ (\Delta, \emptyset) \nVdash_{\mathrm{r}} A.$$

$\underline{\Diamond A \in \Gamma \implies (\Gamma, \Theta) \Vdash_{\mathrm{r}} \Diamond A}$: Take an arbitrary Γ', Θ' with

$$(\Gamma, \Theta) \leq (\Gamma', \Theta').$$

Then $\Diamond A \in \Gamma \subseteq \Gamma'$. First we show

$$\Box^{-1}\Gamma', A \nvdash \Theta'.$$

Otherwise, we would have (since $\Box^{-1}\Gamma'$ is a theory and Θ' is a co-theory)

$$\exists B \in \Box^{-1}\Gamma', \exists C \in \Theta', (A, B \vdash C).$$

However, from $A, B \vdash C$ we can infer $\Diamond A, \Box B \vdash \Diamond C$. Since $\Diamond A, \Box B \in \Gamma', \Diamond C$ it holds that $\Gamma' \vdash \Diamond\Theta'$, which contradicts the consistency of $(\Gamma', \Diamond\Theta')$. So $\Box^{-1}\Gamma', A \nvdash \Theta'$, hence using extension lemma, we obtain some $\Delta \supseteq \Box^{-1}\Gamma' \cup \{A\}$ with $\Delta \nvdash \Theta'$. Then we have

$$(\Gamma', \Theta')\ R\ (\Delta, \emptyset),$$

and since $A \in \Delta$ it holds that $(\Delta, \emptyset) \Vdash_{\mathrm{r}} A$.

$\underline{\Diamond A \notin \Gamma \implies (\Gamma, \Theta) \nVdash_{\mathrm{r}} \Diamond A}$: Assume $\Diamond A \notin \Gamma$, and let

$$\Theta_0 = \{B \mid B \vdash A\}.$$

Then Θ_0 is a co-theory such that (Γ, Θ_0) is consistent. Moreover, we have

$$(\Gamma, \Theta) \leq (\Gamma, \Theta_0).$$

Consider any (Δ, Λ) such that $(\Gamma, \Theta_0)\ R\ (\Delta, \Lambda)$. Then, since (Δ, Θ_0) is consistent and $A \in \Theta_0$, we have $A \notin \Delta$. Therefore $(\Gamma, \Theta) \nVdash_{\mathrm{r}} \Diamond A$. $\Box$

Completeness easily follows from this lemma.

2.5.3 Axiomatization of Non-Fallible Semantics

We have the following completeness theorem for non-fallible version, which shows that $\mathrm{IK}^- + \mathbf{N}_\Diamond$ is complete for non-fallible IR-frame semantics.

Theorem 2.5.13. *For any formula A, the following are equivalent:*

1. *A is a theorem of $\mathrm{IK}^- + \mathbf{N}_\Diamond$;*
2. *A is valid in every non-fallibe IR-frame.*

To prove this, we basically do the same construction as before. We will briefly describe how to adjust the above discussion to the non-fallible setting.

Definition 2.5.14 (Canonical Frame for $\mathrm{IK}^- + \mathbf{N}_\Diamond$)**.** Let $\langle W, \leq, R, F\rangle$ be the canonical frame for IK^-. Then we define the canonical non-fallible IR-frame to be $\langle W - F, \leq, R\rangle$.

The following variant of the extension lemma can easily be proved from the previous version.

Lemma 2.5.15. *In Lemma 2.5.8, if Γ is consistent, then we can choose Γ' so that Γ' is also consistent.*

Proof. If $\Theta \neq \emptyset$, the result is trivial. So consider the case $\Theta = \emptyset$. Since $(\Gamma, \emptyset)$ is always consistent, what we have to prove is: if Γ is consistent, then it has a consistent prime extension. This is immediate from the original extension lemma, because the consistency of Γ is equivalent to the consistency of $(\Gamma, \{\bot\})$. $\Box$

We can prove non-fallible version of Lemma 2.5.12.

Lemma 2.5.16. *Let $\mathcal{R} = \langle W, \leq, R\rangle$ be the canonical frame defined above, and define V in the same way as Lemma 2.5.12. Then it holds that*

$$\mathcal{R}, V, (\Gamma, \Theta) \Vdash_{\mathrm{r}} A \iff A \in \Gamma$$

for every formula A.

Proof. The proof is almost the same as Lemma 2.5.12. We use the above variant of the extension lemma instead of the original one. Only exception is the case of $\Diamond A \in \Gamma \implies (\Gamma, \Theta) \Vdash_{\mathrm{r}} \Diamond A$. Here we have to check $\Box^{-1}\Gamma'$ is consistent to ensure that $(\Delta, \emptyset) \in W$ (that is, Δ is consistent). This is indeed the case since we can show that

$$\bot \in \Box^{-1}\Gamma' \implies \bot \in \Gamma'$$

under the assumption that $\Diamond A \in \Gamma'$, as follows. Suppose $\bot \in \Box^{-1}\Gamma'$. Then we have $\Box\bot \in \Gamma'$ and $\Diamond A \in \Gamma \subseteq \Gamma'$. However, we can also show that $\Box\bot, \Diamond A \vdash_{\mathrm{IK}^- + \mathbf{N}_\Diamond} \bot$ as follows:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\mathbf{N}_\Diamond$}
\UnaryInfC{$\neg\Diamond\bot$}
\AxiomC{}
\RightLabel{$\mathbf{K}_\Diamond$}
\UnaryInfC{$\Box(A \to \bot) \to \Diamond A \to \Diamond\bot$}
\AxiomC{$\Box\bot$}
\noLine
\RightLabel{$\Box$ is monotone}
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Box(A \to \bot)$}
\BinaryInfC{$\Diamond A \to \Diamond\bot$}
\AxiomC{$\Diamond A$}
\BinaryInfC{$\Diamond\bot$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

This means that $\bot \in \Gamma'$, as required. $\Box$

Remark 2.5.17. It seems quite natural that non-fallible semantics can be axiomatized by adding an extra axiom $\mathbf{N}_\Diamond$, because this axiom intuitively says that "there is no inconsistent world," which is the same as to say "there is no fallible world."

2.5.4 Completeness for IM-Models

In the case of IM-models, we have the following axiomatization.

Theorem 2.5.18. *For any formula A, the following are equivalent:*

1. *A is a theorem of* $\mathrm{IK}^- + \mathbf{Dual}$*;*
2. *A is valid in every (non-fallibe) IM-frame.*

The proof is easier than the case of IR-model, mainly because we do not need to care $\Diamond$ under the presence of **Dual**, which makes $\Diamond$ a definable connective.

Definition 2.5.19 (Canonical Frame for $\mathrm{IK}^- + \mathbf{Dual}$)**.** We define the canonical frame $\langle W, \leq, R\rangle$ as follows:

- W is the set of all consistent prime theories of $\mathrm{IK}^- + \mathbf{Dual}$.
- $\leq$ is given by the set-inclusion.
- $\Gamma \, R \, \Gamma'$ if and only if $\Box^{-1}\Gamma \subseteq \Gamma'$.

Then $\langle W, \leq, R\rangle$ is indeed an IM-frame, and the completeness is proved in the same way as the case of IR-frame. Note that, in the proof of Lemma 2.5.12, the second component of a pair (Γ, Θ) plays a significant role only in the cases of $\Diamond$.

2.6 Summary and Remarks

2.6.1 Summary

In this chapter we have introduced a Hilbert-style proof system and Kripke-style models for IMLs. We have considered two types of models, fallible and non-fallible models. Fallible version can be axiomatized by IK^-, and non-fallible version is axiomatized by adding another axiom $\mathbf{N}_\Diamond$ to the basic system IK^-. We have also considered a certain subclass of non-fallible models, called IM-models. This class validates the classical principle $\Diamond p \leftrightarrow \neg\Box\neg p$, the duality of $\Diamond$ and $\Box$, and adding this principle to IK^- is sufficient to axiomatize the class of all IM-models.

2.6.2 Related Work

The canonical model construction presented in this chapter is closely related to the completeness proof by Alechina et al. [4] and the duality theory by Hilken [20].

Alechina et al. proved the completeness of their constructive S4. Their Kripke semantics is basically the same as our fallible IR-semantics (except that they imposed some additional conditions to adapt their semantics to S4 modal logic). Their proof of completeness also uses a canonical model construction similar to ours. Aside from several differences of representation, the only differences between their construction and ours is that they required $\Theta \subseteq \Theta'$ as a condition for $(\Gamma, \Theta)\ R\ (\Gamma', \Theta')$, whereas we used $\Gamma' \cap \Theta = \emptyset$. Actually, under the presence of the axiom $p \to \Diamond p$ (in particular, in S4) the former implies the latter.

Hilken studied Stone duality for intuitionistic modal algebras (Heyting algebras equipped with modal operators $\Box$ and $\Diamond$). He defined the notion of points of modal algebras as a pair of ordinary point (a completely prime filter) p and an element a of the algebra such that $\Diamond a \notin p$.[2] Intuitively a represents (the interior of) the set of points that are not accessible from the point (p, a). Without this extra component, we cannot recover the operation $\Diamond$ from a topological space. By using this notion of points, Hilken proved that we can establish an analogue of Stone duality for intuitionistic modal algebras. Our construction of canonical frame can be understood as another representation of this idea.

[2] Precisely speaking, Hilken calls such a pair "pre-point," and defines the set of points as a certain subset of the set of all pre-points.

Chapter 3

Intuitionistic LTL

3.1 Introduction

3.1.1 Background and Overview

Temporal logic is a family of (modal) logics in which the truth of propositions depends on time, and is useful to describe various properties of state transition systems. Linear-time temporal logic (LTL, for short) is a temporal logic in which each time (state) has a unique successor. LTL is often used to reason about properties of a fixed execution path (an infinite sequence of possible transitions) of a state transition system.

In this chapter, we study a constructive propositional LTL with only the "next" temporal operator denoted by $\bigcirc$. Our main results are (1) to give natural deduction, sequent calculus (satisfying cut elimination), and Hilbert-style proof systems and (2) to give Kripke-style semantics together with a completeness theorem.

The characteristic feature of our version of LTL is that the "distributivity law" $\bigcirc(A \vee B) \to \bigcirc A \vee \bigcirc B$, is *not* admitted. This law is a theorem in the classical LTL [39], and is also admitted in intuitionistic LTL previously considered, such as the ones by Ewald [15] and by Maier [24]. To our knowledge, this is the first version of intuitionistic LTL without the distributivity law.

The motivation not to admit the distributivity law comes from the type-theoretic interpretation of the $\bigcirc$ operator, first given by Davies [13]. He pointed out that a proof system of LTL can be related to a type system of (multi-level) binding-time analysis, which is used in offline partial evaluation to determine which part of a program can be computed at specialization-time and which is residualized. Davies defined a natural deduction system for a constructive LTL with only the "next" operator $\bigcirc$ and implication, and derived via the Curry-Howard isomorphism a typed λ-calculus $\lambda^{\bigcirc}$, which was formally shown to be equivalent to a type system of multi-level binding-time analysis by Glück and Jørgensen [19].

According to his theory, a formula $\bigcirc A$, which logically means that A holds at the next time, is interpreted as a type of (residual) *code* of type A; introduction and elimination rules of $\bigcirc$ are as Lisp-like quasiquote and unquote, respectively. As a result, $\lambda^{\bigcirc}$ terms can be considered as program-generating programs, such as parser generators or generating extensions, which manipulate code fragments by the quasiquotation mechanism. For example, a parser generator would have a type like $\texttt{parser\_spec} \to \bigcirc(\texttt{string} \to \texttt{syntax\_tree})$.

Now, let us consider the type-theoretic reading of the distributivity law. A proof of the distributivity law would be considered a function which takes a value of type $\bigcirc(A \vee B)$ and returns a value of type $\bigcirc A \vee \bigcirc B$. While a value of the return type must be of type $\bigcirc A$ or type $\bigcirc B$ with a tag indicating which of the two is actually the case, a value of the argument type is *quoted* code, which will not be executed *until the next time comes*, that is, until the residual code is executed; it is in general impossible to know which value (A or B) this code evaluates to *now* (unless a Lisp-like eval function was available). From this observation, we conclude that there is no method to turn a value of type $\bigcirc(A \vee B)$ into a value of type $\bigcirc A \vee \bigcirc B$, and hence $\bigcirc A \vee \bigcirc B$ should be strictly stronger than $\bigcirc(A \vee B)$.

Similarly, we also reject $\bigcirc\bot \to \bot$, which is admitted in classical LTL. The falsehood $\bot$ is interpreted as a type which has no value, so a program of type $\bot$ does not terminate normally. However, a program of type $\bigcirc\bot$ can terminate normally, although the resulting value (which is code of type $\bot$) would not, when executed.

If we rephrase these observations in terms of logic, it would be expressed as follows. $\bigcirc(A \vee B)$ is the assertion that "either A or B will hold at the next time." This does not necessarily mean that we can specify at the current time the disjunct that will be true; in general, it can be decided only at the next time. On the other hand, intuitionistically $\bigcirc A \vee \bigcirc B$ means either $\bigcirc A$ or $\bigcirc B$ is true. So when one makes this assertion, the disjunct (that is, either $\bigcirc A$ or $\bigcirc B$) must be specified at the time of assertion (that is, the current time). This explains why $\bigcirc(A \vee B)$ does not imply $\bigcirc A \vee \bigcirc B$.

Similarly, even if we know that the situation will be inconsistent at the next time, it does not necessarily mean that the current situation is also inconsistent. So $\bigcirc\bot$ (inconsistency at the next time) does not imply $\bot$ (inconsistency at the current time).

In short, the disjunction and falsehood at the next time and at the current time are distinguished. This distinction leads us to the rejection of the laws $\bigcirc(A \vee B) \to \bigcirc A \vee \bigcirc B$ and $\bigcirc\bot \to \bot$. We will discuss a related issue later (see Chapter 6).

3.1.2 Organization of the Chapter

The organization of the rest of this chapter is as follows. In Section 3.2, we discuss natural deduction and Hilbert-style proof systems. We first review the system by Davies, and extend it with conjunction, disjunction and falsehood. After that we introduce a Hilbert-style proof system which is equivalent to the natural deduction. In section 3.3 we consider Kripke semantics. It turns out that, although our logic is considered to be a version of LTL, a straightforward extension of classical semantics is not suitable for our interpretation of $\bigcirc$. Having shown the problem, we introduce a suitable Kripke semantics and prove soundness and completeness theorems. In Section 3.4 we define a sequent calculus $\mathrm{LJ}^{\bigcirc}$, which is equivalent to the natural deduction, with cut elimination theorem. Although proof-theoretic consideration is not the main subject of this thesis, the proof of the cut-elimination theorem is worth mentioning, since a difficulty arises from the rejection of the distributivity law, which is our main interest in this chapter. Finally, in Section 3.5, we briefly summarize the chapter, and discuss related work. A brief remark on algebraic semantics is also made.

3.2 Proof Systems for intuitionistic LTL

In this section, we first recall the natural deduction system by Davies and some of its properties, and then extend the system with conjunction, disjunction and falsehood.

3.2.1 Results by Davies

The temporal logic Davies considered contains only $\bigcirc$ ("next" operator) and $\to$ (intuitionistic implication). So here we consider formulas containing only these two connectives.

A judgment in his system takes the form

$$A_1^{n_1}, \ldots, A_k^{n_k} \vdash B^m$$

where A_i, B are formulas and n_i, m are natural numbers; it is read "B holds at time m under the assumption that A_i holds at time n_i (for $i = 1, \ldots, k$)." In what follows, we use A, B, C, D for formulas, k, l, m, n for natural numbers, F, G for annotated formulas (formulas with time annotation) A^n, and Γ, Δ for sets of annotated formulas. We consider the left-hand side of a judgment a set. The notation $\bigcirc^n A$ denotes the formula A applied by n $\bigcirc$'s.

Inference rules of Davies' system are listed in Figure 3.1. The rules $\to$I, $\to$E, and Axiom are standard. The other two, the introduction and elimination rules for $\bigcirc$ operator, state that A holds at time $n+1$ if and only if $\bigcirc A$ holds at time n. This is quite natural since $\bigcirc A$ means that "A holds at the next time."

To show that $\bigcirc$ operator in this system is indeed the "next" operator in linear-time temporal logic, Davies compared his system with $L^{\bigcirc}$, a well-known Hilbert-style proof system of the fragment of classical linear-time temporal logic consisting of only implication, negation and next operators. The axiomatization is given by Stirling, who also proved that $L^{\bigcirc}$ is sound and complete for the standard semantics [39]. The axioms and rules of $L^{\bigcirc}$ are as follows:

$$\frac{}{\Gamma, A^n \vdash A^n} \text{(Axiom)}$$

$$\frac{\Gamma, A^n \vdash B^n}{\Gamma \vdash (A \to B)^n} (\to\text{I})$$

$$\frac{\Gamma \vdash (A \to B)^n \qquad \Gamma \vdash A^n}{\Gamma \vdash B^n} (\to\text{E})$$

$$\frac{\Gamma \vdash A^{n+1}}{\Gamma \vdash (\bigcirc A)^n} (\bigcirc\text{I})$$

$$\frac{\Gamma \vdash (\bigcirc A)^n}{\Gamma \vdash A^{n+1}} (\bigcirc\text{E})$$

Figure 3.1: Derivation Rules of Davies' System.

$$\frac{\Gamma \vdash (A \wedge B)^n}{\Gamma \vdash A^n} (\wedge\text{E1}) \qquad \frac{\Gamma \vdash A^n \qquad \Gamma \vdash B^n}{\Gamma \vdash (A \wedge B)^n} (\wedge\text{I})$$

$$\frac{\Gamma \vdash (A \wedge B)^n}{\Gamma \vdash B^n} (\wedge\text{E2}) \qquad \frac{\Gamma \vdash A^n}{\Gamma \vdash (A \vee B)^n} (\vee\text{I1})$$

$$\frac{\Gamma \vdash (A \vee B)^n \qquad \Gamma, A^n \vdash C^n \qquad \Gamma, B^n \vdash C^n}{\Gamma \vdash C^n} (\vee\text{E}) \qquad \frac{\Gamma \vdash B^n}{\Gamma \vdash (A \vee B)^n} (\vee\text{I2})$$

$$\frac{\Gamma \vdash \perp^n}{\Gamma \vdash A^n} (\perp\text{E})$$

Figure 3.2: Additional Rules for Full NJ$^\bigcirc$.

Axioms

- any classical tautology instance
- $\bigcirc\neg A \to \neg\bigcirc A$
- $\neg\bigcirc A \to \bigcirc\neg A$
- $\bigcirc(A \to B) \to \bigcirc A \to \bigcirc B$

Rules

- if $A \to B$ and A are theorems, then so is B
- if A is a theorem, then so is $\bigcirc A$

Davies extended his system by adding negation with its classical rule (which reads $\Gamma, (\neg A)^n \vdash A^n$ implies $\Gamma \vdash A^n$), and proved that the extended version is equivalent to $L^\bigcirc$ in the following sense [13]:

Proposition 3.2.1. *A judgment $A_1^{n_1}, \ldots, A_k^{n_k} \vdash B^m$ is provable in the extended system if and only if $\bigcirc^{n_1} A_1 \to \ldots \to \bigcirc^{n_k} A_k \to \bigcirc^m B$ has a proof in $L^\bigcirc$. In particular, $\cdot \vdash A^0$ is provable if and only if A is a theorem of $L^\bigcirc$.*

3.2.2 Natural Deduction for Full System

Next we extend Davies' system with conjunction, disjunction and falsehood. We call the extended system NJ$^\bigcirc$. Additional derivation rules are listed in Figure 3.2. The rules for conjunction and introduction rules for disjunction are fairly straightforward, but the other two rules would require some explanation.

In $\vee$E, the formula being eliminated must have the same time as the succedent of the conclusion. At first sight it may seem strange, but in fact this restriction is essential for our system. Indeed, without this restriction we could prove the distributivity law $\bigcirc(A \vee B) \to \bigcirc A \vee \bigcirc B$, which should not be a tautology as mentioned above, as follows:

$$\dfrac{\dfrac{(\bigcirc(A \vee B))^0 \vdash (\bigcirc(A \vee B))^0}{(\bigcirc(A \vee B))^0 \vdash (A \vee B)^1} \quad \dfrac{\dfrac{(\bigcirc(A \vee B))^0, A^1 \vdash A^1}{(\bigcirc(A \vee B))^0, A^1 \vdash (\bigcirc A)^0}}{(\bigcirc(A \vee B))^0, A^1 \vdash (\bigcirc A \vee \bigcirc B)^0} \quad \dfrac{\dfrac{(\bigcirc(A \vee B))^0, B^1 \vdash B^1}{(\bigcirc(A \vee B))^0, B^1 \vdash (\bigcirc B)^0}}{(\bigcirc(A \vee B))^0, B^1 \vdash (\bigcirc A \vee \bigcirc B)^0}}{(\bigcirc(A \vee B))^0 \vdash (\bigcirc A \vee \bigcirc B)^0} \vee\text{E}$$

In this proof, disjunction being eliminated has time 1 while the time of the succedent is 0.[1]

For the same reason, we need to restrict the time of A in $\perp$E to be the same as the time of $\perp$ being eliminated. Otherwise, $\bigcirc\perp \to \perp$ would be a theorem.

[1]For a formal proof that NJ$^\bigcirc$ does not prove the distributivity law, see Theorem 3.4.7.

In fact, the problem would occur only if we allowed the time of the succedent to be strictly less than that of the formula being eliminated. Indeed, a slight variation of $\vee$E in which C^n is changed to C^m with the side condition $m \geq n$ is provable in $\mathrm{NJ}^{\bigcirc}$ by using $\bigcirc$I and $\bigcirc$E. In the same way, a variant of $\perp$E which derives A^m from $\perp^n$ for $m \geq n$ is also provable in $\mathrm{NJ}^{\bigcirc}$.

3.2.3 Hilbert-Style Axiomatization

Next we briefly describe how the logic defined above is characterized in the Hilbert-style. Interestingly, there exists a quite simple axiomatization.

Proposition 3.2.2. *Consider the proof system given by the following sets of axioms and rules.*

Axioms

- *any intuitionistic tautology instance*
- $\bigcirc(A \to B) \to \bigcirc A \to \bigcirc B$
- $(\bigcirc A \to \bigcirc B) \to \bigcirc(A \to B)$

Rules

- *if* $A \to B$ *and* A *are theorems, then so is* B
- *if* A *is a theorem, then so is* $\bigcirc A$

Then, this system is equivalent to $\mathrm{NJ}^{\bigcirc}$ *in the same sense as the Proposition 3.2.1.*

Therefore we can say that our logic, formalized as $\mathrm{NJ}^{\bigcirc}$, is obtained by adding axiom $(\bigcirc A \to \bigcirc B) \to \bigcirc(A \to B)$ to the minimal normal IML (with only $\Box$ modality). We call this axiom **CK** as it is the "converse" of the axiom **K**,

The axiomatization above (in particular, the axiom **CK**) is due to Yuse and Igarashi [45]. They extended Davies' natural deduction system and $\lambda^{\bigcirc}$ with $\Box$ operator, which is similar to "always" operator in classical LTL, and conjectured that their Hilbert-style system and natural deduction system are equivalent. The axiomatization above is its $\Box$-free fragment.

Below we are going to sketch the proof of the proposition. First, we show that the axioms and rules are sound with respect to $\mathrm{NJ}^{\bigcirc}$. The axiom **CK** is the only non-standard clause, so we only check this axiom. Provability of **CK** is easily seen from the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\bigcirc A \to \bigcirc B)^0, A^1 \vdash (\bigcirc A \to \bigcirc B)^0 \qquad \dfrac{(\bigcirc A \to \bigcirc B)^0, A^1 \vdash A^1}{(\bigcirc A \to \bigcirc B)^0, A^1 \vdash (\bigcirc A)^0}\bigcirc\mathrm{I}}{(\bigcirc A \to \bigcirc B)^0, A^1 \vdash (\bigcirc B)^0}\to\mathrm{E}}{(\bigcirc A \to \bigcirc B)^0, A^1 \vdash B^1}\bigcirc\mathrm{E}}{(\bigcirc A \to \bigcirc B)^0 \vdash (A \to B)^1}\to\mathrm{I}}{(\bigcirc A \to \bigcirc B)^0 \vdash (\bigcirc(A \to B))^0}\bigcirc\mathrm{I}}{\cdot \vdash ((\bigcirc A \to \bigcirc B) \to \bigcirc(A \to B))^0}\to\mathrm{I}$$

For the converse, we only mention the admissibility of $\to$I, which is the most essential part (actually $\vee$E is also nontrivial, but can be checked in a similar way). Putting Γ aside, this rule says that "if $A^n \vdash B^n$, then $\cdot \vdash (A \to B)^n$." To prove this rule is admissible, it is sufficient to show that "if $\bigcirc^n A \to \bigcirc^n B$, then $\bigcirc^n(A \to B)$," and this is an immediate consequence of the axiom **CK**.

3.3 Kripke Semantics

In this section, we consider Kripke semantics for the logic defined above, and establish soundness and completeness for that semantics.

Before going into the technical details, we sketch the rest of this section. First we mention that classical LTL can be described in terms of Kripke frames whose accessibility relation is a function. From this fact it seems natural to consider a semantics based on birelational semantics whose modal accessibility is a function. Unfortunately, however, exploiting this condition turns out to be inappropriate, because the resulting semantics admits the distributivity law, which we need to avoid. After seeing that, we examine IM-frames (Definition 2.4.12). We can give a class of IM-frames which corresponds to our logic, by identifying the corresponding frame condition. This approach works well in the sense that it establishes a semantics for which soundness and completeness hold. However, it this semantics the linearity of time has been lost. Moreover, the meaning of the frame condition is not intuitively understandable. For this reason, we consider another version from this semantics. This is achieved by decomposing modal accessibility relation of IM-frames (actually, this decomposition process appears implicitly in the proof of completeness). This derives another class of birelational frames whose modal accessibility is a partial function with some properties. As a result we obtain an equivalent, more comprehensible semantics for our logic.

3.3.1 Functional Kripke Frames

In this subsection, we are going to examine a class of birelational Kripke frames which comes from the semantics of classical LTL in a fairly straightforward manner. Although this semantics seems natural, and works well for the implicational fragment, it turns out that it admits the distributivity law.

Consider Kripke frames whose accessibility relation R on possible worlds is a function. Such frames are said to be *functional*. The terminology "functional frame" is, to our knowledge, first used by Segerberg [35] (to be precise, he used the terminology "totally functional frames" to mean functional frames in our terminology), but not in the context of the semantics of LTL. In a functional Kripke frame, the next state of a given state is uniquely determined, hence it justifies "linear time." Although the semantics of classical LTL is often given by using execution paths of transition systems, it is easy to translate it into Kripke-style semantics with functional frames.

Now, let us consider functional frames augmented by an intuitionistic accessibility relation $\leq$.

Definition 3.3.1. An *intuitionistic functional frame* is a triple $\langle W, \leq, R\rangle$ of a nonempty set W, a partial order $\leq$ on W and a function R from W to W such that $(\leq ; R) = (R ; \leq)$ holds.

This notion is an extension of classical functional frames: if $\leq$ is the diagonal relation (that is, $x \leq y$ if and only if $x = y$) in this definition, the frame $\langle W, \leq, R\rangle$ can be identified with a classical functional frame $\langle W, R\rangle$. In what follows, we simply say functional frame when no confusion arises.

Since a functional frame is an IR-frame defined in Section 2.4, it naturally defines an interpretation of modal formulas in it. Notions of a valuation and its admissibility is given as in Definition 2.4.2, and satisfaction relation is given as in Definition 2.4.4, regarding $\bigcirc$ as $\Box$.

Notation. We also write $\mathcal{F}, V, w \Vdash A^n$ for $\mathcal{F}, V, w \Vdash \bigcirc^n A$, and omit $\mathcal{F}$ or V if they are clear from the context.

It is not very difficult to see that soundness and completeness hold for the $\vee, \bot$-free fragment. Soundness is proved by straightforward induction on the derivation. Completeness is proved by constructing a canonical model.

Canonical model is constructed as follows. For a set T of formulas, we define

$$\bigcirc^{-1}T := \{A \mid \bigcirc A \in T\}\,;$$
$$\bigcirc T := \{\bigcirc A \mid A \in T\}\,.$$

The canonical frame $\langle W, \leq, R\rangle$ and the canonical valuation V is given by

- W is the set of all theories (sets of formulas closed under deduction) of $\vee, \bot$-free fragment,
- $\leq$ is the set-inclusion,
- $R = \bigcirc^{-1}$, and
- $V(p) = \{T \in W \mid p \in T\}$.

It it easy to show that $\langle W, \leq, R\rangle$ is a functional frame and V is admissible. As usual, it holds that $T \Vdash A \iff A \in T$ for each formula A. Finally, if $\Gamma \vdash A^n$ is not provable, take the set $\left\{A \,\middle|\, \Gamma \vdash A^0\right\}$ as T. Then $T \Vdash \Gamma$ holds but $T \Vdash A^n$ does not.

The proof strategy above is almost standard, but notice that we took the set of all theories as W, instead of taking only prime theories. When we consider disjunction and falsehood, the same method will not work. In fact, functional frames are not appropriate in the presence of these connectives, because they validate the laws $\bigcirc(A \vee B) \to \bigcirc A \vee \bigcirc B$ and $\bigcirc\bot \to \bot$, which we have rejected. It does not seem easy to adjust the definition of the satisfaction relation to exclude them without relaxing the functionality condition.

3.3.2 Semantics Based on IM-frames

In this subsection, we put functionality aside and consider a large class of frames, and try to find its subclass corresponding to the intended logic.

As discussed above, the logic defined by $\mathrm{NJ}^{\bigcirc}$ is defined so that it does not admit the distributivity, so $\mathrm{NJ}^{\bigcirc}$ is an appropriate logic for our motivation. If the completeness for the full system fails, it means that the choice of functional frame was incorrect. Therefore the next question is what kind of frames correspond to our logic.

The first answer we give is $\mathrm{IM}^{\bigcirc}$-frames defined below.

Definition 3.3.2. An *$IM^{\bigcirc}$-frame* is an IM-frame $\langle W, \leq, R\rangle$ satisfying the following condition: if $w\ R\ v$, then there exists w' such that $w \leq w'$ and $\forall u \in W.(w'\ R\ u \iff v \leq u)$.

Note that, in the definition of $\mathrm{IM}^{\bigcirc}$-frame above, R is not assumed to be a function.

The satisfaction relation is defined in the same way as in Section 2.4 (again regarding $\bigcirc$ as $\Box$), and heredity is also verified easily.

Theorem 3.3.3 (Soundness). *Suppose that $\Gamma \vdash A^n$ is provable in $\mathrm{NJ}^{\bigcirc}$. Then for any $IM^{\bigcirc}$-frame $\mathcal{F} = \langle W, \leq, R\rangle$, an admissible valuation V, and a possible world $w \in W$, if $\mathcal{F}, V, w \Vdash \Gamma$ then $\mathcal{F}, V, w \Vdash A^n$.*

Proof. Induction on the derivation. □

Theorem 3.3.4 (Completeness). *If $w \Vdash \Gamma$ implies $\mathcal{F}, V, w \Vdash A^n$ for any $IM^{\bigcirc}$-frame $\mathcal{F} = \langle W, \leq, R\rangle$, an admissible valuation V, and possible world $w \in W$, then there exists a derivation of $\Gamma \vdash A^n$.*

To prove this, we use the canonical model construction defined in Definition 2.5.19. Let $\langle W, \leq, R\rangle$ be the canonical model. Then it is easy to see that the following hold.

1. $\langle W, \leq, R\rangle$ forms an IM-frame.
2. Let V be the canonical valuation defined by $V(p) = \{T \mid p \in T\}$. Then $T \Vdash A \iff A \in T$ holds for each formula A.

Therefore, we only need to check that the canonical frame is an $\mathrm{IM}^{\bigcirc}$-frame. Below we check that it satisfies the condition of Definition 3.3.2.

Lemma 3.3.5. *Let $S, T \in W$. Then, $\forall X \in W(T\ R\ X \iff S \subseteq X)$ if and only if $\bigcirc^{-1}T = S$.*

Proof. The right-to-left direction is obvious. To prove the other direction by contraposition, assume $\bigcirc^{-1}T \neq S$. Then we have either $\bigcirc^{-1}T \not\subseteq S$ or $S \not\subseteq \bigcirc^{-1}T$. In the first case, $T\ R\ X \iff S \subseteq X$ does not hold when $X = S$. In the second case, there exists some formula A such that $A \in S$ and $A \notin \bigcirc^{-1}T$. Then, in the usual way we can prove that there exists a prime theory V such that $\bigcirc^{-1}T \subseteq V$ and $A \notin V$ (therefore $T\ R\ V$ but $S \not\subseteq V$). □

Lemma 3.3.6. *For $S, T \in W$ such that $S\ R\ T$, there exists a theory U (not necessarily prime) satisfying $\bigcirc^{-1}U = S$ and $T \subseteq U$.*

Proof. Let U be the set of all formulas provable from T and $\bigcirc S$. First, we check that U is a theory. It is clear that U is deductively closed. To check U is consistent, suppose $\bot \in U$. Then we have $\bigcirc\bot \in U$, hence $\bot \in \bigcirc^{-1}U = S$, a contradiction.

We are going to prove that $\bigcirc^{-1}U = S$ and $T \subseteq U$ hold for this U. Clearly, $T \subseteq U$ holds by definition. It is also easy to see that $S \subseteq \bigcirc^{-1}U$: if $A \in S$, then $\bigcirc A \in \bigcirc S \subseteq U$, and from this $A \in \bigcirc^{-1}U$ follows. For the converse, let A be a formula in $\bigcirc^{-1}U$. Then we have $\bigcirc A \in U$. Since U is the smallest theory containing T and $\bigcirc S$, there exist formulas $A_1, \ldots, A_n \in S$ ($n \geq 0$) such that $\bigcirc A_1 \to \ldots \to \bigcirc A_n \to \bigcirc A \in T$. Then, since axiom **CK** is provable, we also have $\bigcirc(A_1 \to \ldots \to A_n \to A) \in T$. This implies that $A_1 \to \ldots \to A_n \to A \in \bigcirc^{-1}T \subseteq S$ holds. As $A_i \in S$ from the assumption, we conclude that $A \in S$, as required. □

Lemma 3.3.7. *Let $S, T \in W$ such that $S\ R\ T$. Then, any maximal element of*

$$X = \left\{U \;\middle|\; U \text{ is a theory such that } \bigcirc^{-1}U = S \text{ and } T \subseteq U\right\}.$$

is prime.

Proof. Let $U \in X$ be a maximal element and suppose $A_1, A_2 \notin U$. Moreover, let U_0, U_1, U_2 be the smallest theory extending U with $A_1 \vee A_2, A_1, A_2$, respectively. It is sufficient to prove that $U_0 \neq U$.

For $i = 1, 2$ the theory $\bigcirc^{-1} U_i$ is a proper extension of $\bigcirc^{-1} U = S$, so there exists a formula $B_i \in \bigcirc^{-1} U_i \setminus S$. For such B_1 and B_2, it holds that $\bigcirc(B_1 \vee B_2) \in U_1 \cap U_2 = U_0$ and $B_1 \vee B_2 \notin S = \bigcirc^{-1} U$ (because S is prime). Therefore we obtain $\bigcirc(B_1 \vee B_2) \in U_0 \setminus U$, and this implies $U_0 \neq U$, as required. □

Putting these lemmas together, we can see that the canonical frame defined above is indeed an IM$^{\bigcirc}$-frame, from which the completeness follows.

Finally, let us mention a relationship between IM$^{\bigcirc}$-frame and functional frame. There exists a translation from functional into IM$^{\bigcirc}$-frame, given as follows:

Proposition 3.3.8. *For an arbitrary functional frame $\mathcal{F} = \langle W, \leq, R \rangle$, consider the binary relation $R' = (R \,;\, \leq)$. Then the frame $\mathcal{F}' = \langle W, \leq, R' \rangle$ is an IM$^{\bigcirc}$-frame, and satisfaction relation $\Vdash$ on $\mathcal{F}$ and $\mathcal{F}'$ coincide.*

3.3.3 Partially Functional Kripke Frames

We have established the soundness and completeness theorem, and therefore we can say that IM$^{\bigcirc}$-frames defined above capture our logic. However, while the logic is considered a version of LTL, the condition appearing in the Definition 3.3.2 does not seem to justify the linearity of time. Additionally, the intuitive meaning of the condition is not clear.

In this subsection we try to modify the semantics defined above so that the resulting semantics represents the linear-time nature more directly. We consider another class of birelational Kripke frames, in which each state has at most one next state (although it may have no next state).

Definition 3.3.9. For an IM-frame $\langle W, \leq, R \rangle$, we define another relation R^s by

$$x \; R^s \; y \iff \forall z.(x \; R \; z \iff y \leq z).$$

Then, the condition of Definition 3.3.2 is rephrased by the equality $R = (\leq \,;\, R^s)$. It is easy to check that the following hold:

Lemma 3.3.10. *If $\langle W, \leq, R \rangle$ is an IM$^{\bigcirc}$-frame, then*

1. *R^s is a partial function;*
2. *R^s preserves $\leq$, that is, if $x \; R^s \; y$, $x' \; R^s \; y'$, and $x \leq x'$, then $y \leq y'$;*
3. *$(R^s)^{-1}$ is a simulation relation over $\langle W, \leq \rangle$. In other words, the inclusion $(R^s ; \leq) \subseteq (\leq ; R^s)$ holds.*

This observation motivates the following definition.

Definition 3.3.11. We say an IR-frame (Definition 2.4.1) $\langle W, \leq, S \rangle$ an *intuitionistic partially functional frame* (IPF-frame, for short), if S is a partial function preserving $\leq$. An IPF-frame is said to be an *IPF$^{\bigcirc}$-frame* if S^{-1} is a simulation relation over $\langle W, \leq \rangle$.[2]

From Lemma 3.3.10, for each IM$^{\bigcirc}$-frame $\langle W, \leq, R \rangle$ we can construct an IPF$^{\bigcirc}$-frame $\langle W, \leq, R^s \rangle$ associated to it. We denote this construction by s. Conversely, each IPF$^{\bigcirc}$-frame gives rise to an IM$^{\bigcirc}$-frame $\langle W, \leq, (\leq \,;\, S) \rangle$. It is easy to check that this is indeed an IM$^{\bigcirc}$-frame. We denote the construction of this direction by r. We also use the notation S^r for $(\leq \,;\, S)$.

Moreover, we can show that r is a left-inverse of s. That is, when we construct an IPF$^{\bigcirc}$-frame from an arbitrary IM$^{\bigcirc}$-frame, and transforming it back to an IM$^{\bigcirc}$-frame, then the resulting frame is the same as the original one. This is an easy consequence of the equality $R = (\leq \,;\, R^s)$ mentioned above.

Since an IPF$^{\bigcirc}$-frame is an IR-frame satisfying additional conditions, we can interpret formulas of LTL in it as defined in Definition 2.4.4. Here, as before, we identify $\bigcirc$ as $\Box$, so the truth condition for $\bigcirc$ reads:

$$w \Vdash \bigcirc A \iff \forall w', v.(w \leq w' \; S \; v \implies v \Vdash A).$$

[2] Actually, the condition of S being a partial function is redundant. Any binary relation S satisfying the preservation and simulation conditions is a partial function.

$$\frac{(A \text{ is atomic})}{\Gamma, A^n \Rightarrow A^n}\ (\mathrm{Init}) \qquad \frac{\Gamma \Rightarrow F \qquad F, \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow G}\ (\mathrm{Cut})$$

$$\frac{\Gamma \Rightarrow A^n \qquad \Gamma, B^n \Rightarrow F}{\Gamma, (A \to B)^n \Rightarrow F}\ (\to\mathrm{L}) \qquad \frac{\Gamma, A^n \Rightarrow B^n}{\Gamma \Rightarrow (A \to B)^n}\ (\to\mathrm{R})$$

$$\frac{\Gamma, A^n \Rightarrow F}{\Gamma, (A \wedge B)^n \Rightarrow F}\ (\wedge\mathrm{L1}) \qquad \frac{\Gamma \Rightarrow A^n \qquad \Gamma \Rightarrow B^n}{\Gamma \Rightarrow (A \wedge B)^n}\ (\wedge\mathrm{R})$$

$$\frac{\Gamma, B^n \Rightarrow F}{\Gamma, (A \wedge B)^n \Rightarrow F}\ (\wedge\mathrm{L2}) \qquad \frac{\Gamma \Rightarrow A^n}{\Gamma \Rightarrow (A \vee B)^n}\ (\vee\mathrm{R1})$$

$$\frac{\Gamma, A^n \Rightarrow C^{n+m} \qquad \Gamma, B^n \Rightarrow C^{n+m}}{\Gamma, (A \vee B)^n \Rightarrow C^{n+m}}\ (\vee\mathrm{L}) \qquad \frac{\Gamma \Rightarrow B^n}{\Gamma \Rightarrow (A \vee B)^n}\ (\vee\mathrm{R2})$$

$$\frac{\Gamma, A^{n+1} \Rightarrow F}{\Gamma, (\bigcirc A)^n \Rightarrow F}\ (\bigcirc\mathrm{L}) \qquad \frac{\Gamma \Rightarrow A^{n+1}}{\Gamma \Rightarrow (\bigcirc A)^n}\ (\bigcirc\mathrm{R})$$

$$\frac{(A \text{ is atomic})}{\Gamma, \bot^n \Rightarrow A^{n+m}}\ (\bot\mathrm{L})$$

Figure 3.3: Inference Rules of LJ$^{\bigcirc}$.

Because $w \leq w' \ S \ v$ on the right-hand side is equivalent to $w \ S^r \ v$, we have

$$w \Vdash \bigcirc A \iff \forall v.(w \ S^r \ v \implies v \Vdash A),$$

which is the same as the interpretation in IM-frame obtained by translation r.

Similarly, interpretation in an IM-frame is, since $R = (\leq ; R^s)$,

$$\begin{aligned} w \Vdash \bigcirc A &\iff \forall v.(w \ R \ v \implies v \Vdash A) \\ &\iff \forall w', v.(w \leq w' \ R^s \ v \implies v \Vdash A), \end{aligned}$$

so this is the same as the semantics on the IPF-frame obtained by s.

In this way we can see that two semantics based on IM$^{\bigcirc}$-frames and IPF$^{\bigcirc}$-frames are equivalent. This observation shows that IPF$^{\bigcirc}$-frames are another characterization of our logic.

3.4 Sequent Calculus

In this section we give another formalization LJ$^{\bigcirc}$ of our logic in the sequent calculus style. After verifying that the system LJ$^{\bigcirc}$ is equivalent to NJ$^{\bigcirc}$, we prove the cut-elimination theorem for LJ$^{\bigcirc}$.

3.4.1 Formalization

Sequents of LJ$^{\bigcirc}$ have the form $\Gamma \Rightarrow F$ where Γ is a set of annotated formulas and F is an annotated formula. Inference rules of LJ$^{\bigcirc}$ are listed in Figure 3.3.

Since we regard the left-hand side of a sequent as a set, exchange and contraction rules are not explicitly included. There is no explicit weakening rule, either—we included weakening implicitly by allowing extra formulas in the rules Init and $\bot$L. Most of the rules are standard, but we comment on some rules. In rules Init and $\bot$L, we restricted the right-hand side to be atomic to make the proof of the cut elimination theorem simpler (but this does not reduce the proof-theoretic strength). In rules $\bot$L and $\vee$L, the time of the succedent must be no less than that of the principal formula ($\bot$ and $A \vee B$, respectively). This corresponds to the issue mentioned in Section 3.2 that we cannot eliminate falsehood or disjunction with a succedent of an earlier time.

In $\vee$L and $\bot$L, m may be positive. This means that the time of the succedent is not required to be the same as that of principal formula. If these rules were applicable only when the two time annotations coincide (that is, $m = 0$), then the cut elimination does not hold, although this restriction does not affect proof-theoretic strength under the presence of the cut rule.

Indeed, to derive $(A \vee B)^0, (A \to \bigcirc C)^0, (B \to \bigcirc C)^0 \Rightarrow C^1$ without cut, the last rule must be $\vee$L and in this step the succedent has time 1 while the principal formula has time 0.

$\mathrm{LJ}^{\bigcirc}$ is equivalent to $\mathrm{NJ}^{\bigcirc}$ in the following sense:

Theorem 3.4.1. *A sequent $\Gamma \Rightarrow F$ is provable in $\mathrm{LJ}^{\bigcirc}$ if and only if $\Gamma \vdash F$ is provable in $\mathrm{NJ}^{\bigcirc}$.*

To prove this, it is sufficient to check that all rules of $\mathrm{LJ}^{\bigcirc}$ are admissible in $\mathrm{NJ}^{\bigcirc}$ and vice versa. For the former part we need the admissibility of weakening and cut in natural deduction:

Lemma 3.4.2. *1. If $\Gamma \vdash F$ is provable, then $\Gamma, \Delta \vdash F$ is also provable.*

2. If $\Gamma \vdash F$ and $F, \Delta \vdash G$ are provable, then $\Gamma, \Delta \vdash G$ is also provable.

After checking this lemma, both directions of the theorem are proved by easy induction, so we omit the details.

3.4.2 Cut Elimination

Next we prove the cut elimination theorem by showing that the cut rule is admissible in the cut-free fragment of $\mathrm{LJ}^{\bigcirc}$. The basic strategy of the proof is similar to Pfenning's structural cut elimination [30], although structural cut elimination does not introduce any measure like level or rank used in our proof.

Theorem 3.4.3. *If $\Gamma \Rightarrow F$ and $F, \Delta \Rightarrow G$ are provable without cut, then $\Gamma, \Delta \Rightarrow G$ is also provable without cut.*

We first introduce several auxiliary notions and sketch the proof strategy.

Definition 3.4.4.
1. We simply say a *cut* for any instance of the rule Cut (or for a pair of derivations whose conclusions match the premises of Cut).
2. In a given cut, its *cut formula* is the formula being eliminated (F in the rule Cut of Figure 3.3).
3. The *level* of a cut is the sum of the heights of the derivations of the premises.
4. The *rank* of a formula A, denoted by $|A|$, is defined by: $|A| = 0$ if A is atomic, and $|A \to B| = |A \wedge B| = |A \vee B| = 1 + \max(|A|, |B|)$.
5. The *rank* of a cut is the rank of the cut formula A (without time annotation).

The proof of the theorem is done by induction on the pair of the rank and the level with lexicographic order. In other words, we will prove that any cut is either

1. reduced to a cut with a smaller rank (and possibly with a larger level),
2. reduced to a cut with the same cut formula and a smaller level, or
3. eliminated immediately.

Let us call the derivations of $\Gamma \Rightarrow F$ and $F, \Delta \Rightarrow G$ assumed in the theorem $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, and the rules used in the last steps of them R_1 and R_2.

The proof is done by case analysis on R_1 and R_2, and except for some cases it is rather straightforward to rewrite a given cut into a simpler one. However, a problem stems from the side condition in $\vee$L and $\perp$L. Consider the case $R_1 = \vee$L:

$$\dfrac{\dfrac{\Gamma, A^n \Rightarrow C^m \quad \Gamma, B^n \Rightarrow C^m}{\Gamma, (A \vee B)^n \Rightarrow C^m}\,\vee\mathrm{L} \qquad C^m, \Delta \Rightarrow D^l}{\Gamma, (A \vee B)^n, \Delta \Rightarrow D^l}\,\mathrm{Cut}$$

At first sight, it may seem natural to rewrite this derivation into the following one:

$$\dfrac{\dfrac{\Gamma, A^n \Rightarrow C^m \quad C^m, \Delta \Rightarrow D^l}{\Gamma, A^n, \Delta \Rightarrow D^l}\,\mathrm{Cut} \qquad \dfrac{\Gamma, B^n \Rightarrow C^m \quad C^m, \Delta \Rightarrow D^l}{\Gamma, B^n, \Delta \Rightarrow D^l}\,\mathrm{Cut}}{\Gamma, (A \vee B)^n, \Delta \Rightarrow D^l}\,\vee\mathrm{L}$$

However, this is not always a valid derivation, because it is not necessarily the case that $l \geq n$.

By case analysis on R_2 we can see that the problem is essential when the cut formula C is neither atomic formula nor disjunction (details are presented below). In these cases we need to transform $\mathcal{D}_1$ appropriately before reducing the cut. To this end, we use the following lemma.

Lemma 3.4.5. *If a sequent $S \equiv \Gamma \Rightarrow F$ has a cut-free derivation $\mathcal{D}$ and F is neither atomic formula nor disjunction, then there exists a cut-free derivation $\mathcal{D}'$ of S such that the last rule used in $\mathcal{D}'$ is a right rule.*

Proof. From the assumption that F is neither atomic formula nor disjunction, the derivation ends with either $\to$R, $\wedge$R, or $\bigcirc$R followed by some (possibly zero) left rules. Note that neither Init nor $\bot$L can derive $\Gamma \Rightarrow F$ unless F is atomic.

From this observation we can see that it is sufficient to show that these three right rules commute with any left rule following them:

$$\dfrac{\dots \quad \dfrac{T_i}{S_i'}\,\text{Right} \quad \dots}{S}\,\text{Left} \quad \Longrightarrow \quad \dfrac{\dfrac{\dots \quad T_i \quad \dots}{S'}\,\text{Left}}{S}\,\text{Right}$$

This is checked by straightforward case analysis. The point is that rules $\to$R, $\wedge$R, or $\bigcirc$R have only one premise (this is crucial when the left rule is $\vee$L).

For example, if the left rule is $\to$L and the right rule is $\to$R, then

$$\dfrac{\Gamma \Rightarrow A^n \quad \dfrac{\Gamma, B^n, C^m \Rightarrow D^m}{\Gamma, B^n \Rightarrow (C \to D)^m}\,{\to}\text{R}}{\Gamma, (A \to B)^n \Rightarrow (C \to D)^m}\,{\to}\text{L} \quad \Longrightarrow \quad \dfrac{\dfrac{\Gamma \Rightarrow A^n \quad \Gamma, B^n, C^m \Rightarrow D^m}{\Gamma, (A \to B)^n, C^m \Rightarrow D^m}\,{\to}\text{L}}{\Gamma, (A \to B)^n \Rightarrow (C \to D)^m}\,{\to}\text{R}$$

and for $\vee$L and $\bigcirc$R we have

$$\dfrac{\dfrac{\Gamma, A^n \Rightarrow C^{m+1}}{\Gamma, A^n \Rightarrow (\bigcirc C)^m}\,\bigcirc\text{R} \quad \dfrac{\Gamma, B^n \Rightarrow C^{m+1}}{\Gamma, B^n \Rightarrow (\bigcirc C)^m}\,\bigcirc\text{R} \quad (m \geq n)}{\Gamma, (A \vee B)^n \Rightarrow (\bigcirc C)^m}\,\vee\text{L}$$

$$\Longrightarrow \quad \dfrac{\dfrac{\Gamma, A^n \Rightarrow C^{m+1} \quad \Gamma, B^n \Rightarrow C^{m+1} \quad (m+1 \geq n)}{\Gamma, (A \vee B)^n \Rightarrow C^{m+1}}\,\vee\text{L}}{\Gamma, (A \vee B)^n \Rightarrow (\bigcirc C)^m}\,\bigcirc\text{R}$$

Other cases are similar.

After checking this, we can prove the lemma by induction on the number of left rules following the last right rule. □

Proof of Theorem 3.4.3. By lexicographic induction on the rank and the level of a cut. We split the situation into five cases:

1. F is not principal in $\mathcal{D}_2$;
2. F is principal in $\mathcal{D}_2$;
   (a) R_1 is neither $\vee$L nor $\bot$L;
   (b) $R_1 = \bot$L;
   (c) $R_1 = \vee$L, and F is either atomic or disjunction;
   (d) $R_1 = \vee$L, and F is neither atomic nor disjunction.

In case 1, the cut level is easily reduced (without changing the rank); we just lift the cut into $\mathcal{D}_2$. As the cut formula is not principal in $\mathcal{D}_2$, it occurs in all premises of R_2, so this procedure works. For example, if $R_2 = \bigcirc$L we proceed

$$\dfrac{\Gamma \Rightarrow F \quad \dfrac{F, \Delta', A^{n+1} \Rightarrow G}{F, \Delta', \bigcirc A^n \Rightarrow G}\,\bigcirc\text{L}}{\Gamma, \Delta', \bigcirc A^n \Rightarrow G}\,\text{Cut} \quad \Longrightarrow \quad \dfrac{\dfrac{\Gamma \Rightarrow F \quad F, \Delta', A^{n+1} \Rightarrow G}{\Gamma, \Delta', A^{n+1} \Rightarrow G}\,\text{Cut}}{\Gamma, \Delta', \bigcirc A^n \Rightarrow G}\,\bigcirc\text{L}$$

In case 2a, if R_1 is a right rule, we can reduce the cut into cut(s) on subformula(s) of F. For example, if the cut formula is $F = (A \to B)^n$, we rewrite the cut as follows:

$$\dfrac{\dfrac{\Gamma, A^n \Rightarrow B^n}{\Gamma \Rightarrow (A \to B)^n}\to\mathrm{R} \quad \dfrac{\Delta \Rightarrow A^n \quad \Delta, B^n \Rightarrow G}{\Delta, (A \to B)^n \Rightarrow G}\to\mathrm{L}}{\Gamma, \Delta \Rightarrow G}\,\mathrm{Cut}$$

$$\Longrightarrow \dfrac{\dfrac{\Delta \Rightarrow A^n \quad \Gamma, A^n \Rightarrow B^n}{\Gamma, \Delta \Rightarrow B^n}\,\mathrm{Cut} \quad \Delta, B^n \Rightarrow G}{\Gamma, \Delta \Rightarrow G}\,\mathrm{Cut}$$

Since $|A| < |A \to B|$, by the induction hypothesis the cut on A can be eliminated. So $\Gamma, \Delta \Rightarrow B^n$ has a cut-free derivation. Then, since $|B| < |A \to B|$, by the induction hypothesis the cut on B can also be eliminated. This means that $\Gamma, \Delta \Rightarrow G$ has a cut-free proof.

If R_1 is not a right rule, the cut can be lifted into $\mathcal{D}_1$, producing a cut of smaller level (if R_1 is neither $\vee$L nor $\bot$L, this process does not cause the problem with time annotations mentioned above).

In case 2b, from the side condition of $\bot$L, the cut formula F is atomic. So there are only two possibilities: $R_2 = \bot\mathrm{L}$, or $R_2 = \mathrm{Init}$. The second case is easily checked, so we only need to check the first case.

If $R_2 = \bot\mathrm{L}$, the cut has the from

$$\dfrac{\Gamma, \bot^n \Rightarrow \bot^m \quad \bot^m, \Delta \Rightarrow A^l}{\Gamma, \bot^n, \Delta \Rightarrow A^l}\,\mathrm{Cut}$$

where $n \le m \le l$. In this case the conclusion can be derived directly by using $\bot$L because the side condition $n \le l$ is met.

In case 2c, the rule R_2 is Init, $\bot$L, or $\vee$L. The case of Init is obvious. If $R_2 = \vee\mathrm{L}$, derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ have the forms

$$\mathcal{D}_1 = \dfrac{\Gamma, A^n \Rightarrow (C_1 \vee C_2)^m \quad \Gamma, B^n \Rightarrow (C_1 \vee C_2)^m}{\Gamma, A \vee B^n \Rightarrow (C_1 \vee C_2)^m}\,\vee\mathrm{L}$$

$$\mathcal{D}_2 = \dfrac{C_1^m, \Delta \Rightarrow D^l \quad C_2^m, \Delta \Rightarrow D^l}{(C_1 \vee C_2)^m, \Delta \Rightarrow D^l}\,\vee\mathrm{L}$$

and we have $n \le m$ and $m \le l$ from the side condition of $\vee$L. Therefore by lifting $\mathcal{D}_2$ into $\mathcal{D}_1$ we obtain a cut of smaller level. The case of $\bot$L is similar.

The last case is the case 2d, in which F is neither atomic nor disjunction. In this case, using Lemma 3.4.5 we first rewrite $\mathcal{D}_1$ into another derivation $\mathcal{D}_1'$ ending with a right rule. Then, the given cut is transformed into a principal cut, which is easily reduced into a cut of smaller rank (as mentioned in case 2a). □

From the argument above, we obtain the cut-elimination theorem for LJ$^{\bigcirc}$. This can be proved by induction on derivations.

Theorem 3.4.6. *If a sequent is provable in* LJ$^{\bigcirc}$*, then it has a cut-free proof.*

The following is an easy consequence of cut-elimination theorem and equivalence of LJ$^{\bigcirc}$ and NJ$^{\bigcirc}$.

Theorem 3.4.7. *In neither* LJ$^{\bigcirc}$ *nor* NJ$^{\bigcirc}$ *the distributivity law* $\bigcirc(A \vee B) \to \bigcirc A \vee \bigcirc B$ *is provable, as well as* $\bigcirc\bot \to \bot$*.*

This result shows that our systems indeed have an intended property of rejecting these laws.

3.5 Summary and Remarks

3.5.1 Summary

We have investigated a constructive LTL as a logical counterpart of Davies' $\lambda^{\bigcirc}$. We first gave a natural deduction style and Hilbert-style proof systems, and Kripke semantics, together with the proofs of soundness and completeness. We also defined a sequent calculus which enjoys cut elimination.

Although the temporal logic we considered is *linear-time*, a naive frame condition of functionality turned out to be insufficient. We considered two classes of Kripke frames, and gave the connection between two versions of our semantics. We have also discussed relationship between frame conditions and the syntactic counterpart of the logic.

For cut elimination, we basically followed the standard method that lifts non-principal cuts and decompose cut-formulas when the cut is principal. However, to make it work correctly, extra transformations are needed.

3.5.2 Interpreting the Frame Conditions

Above we have proved that our logic is captured by either IM$^{\bigcirc}$-frames or IPF$^{\bigcirc}$-frames, but we did not discuss what their frame conditions mean. In this subsection we discuss the intuitive meaning of IPF$^{\bigcirc}$-frames.

First we consider the condition of IPF$^{\bigcirc}$-frames. On one hand, we have the standard interpretation of Kripke semantics for intuitionistic logic: each possible world represents a state of knowledge, and $\leq$ represents an extension of knowledge. On the other hand, the definition of IPF$^{\bigcirc}$-frames says that S^{-1} should be a simulation. Putting these together, we can say that the condition that S^{-1} is a simulation relation says that any extension of knowledge at the next state can be simulated by some extension of knowledge at the current state. Actually, this appears to be achieved as follows: if we consider the next state extended with new knowledge A, the resulting state would be simulated by adding $\bigcirc A$ to the current state (in fact, we implicitly used this intuitive understanding in the proof of completeness). Therefore, the simulation condition implies that we can indeed identify A^n and $\bigcirc^n A$.

In NJ$^{\bigcirc}$ formalization, the identification between A^n and $\bigcirc^n A$ is justified by rules $\bigcirc$I and $\bigcirc$E. To obtain the equivalence of A^n and $\bigcirc^n A$, it is crucial that NJ$^{\bigcirc}$ does not assume any side condition in $\bigcirc$-rules, which is typically assumed in $\Box$-rules of other systems [37, 25, 14].

From a type-theoretic viewpoint, this corresponds to the fact that $\lambda^{\bigcirc}$ can manipulate open code fragment. Indeed, other type systems based on modal logics assuming side-conditions on $\Box$-rules do not allow code fragments with free variables to be well-typed [25, 14].

In this way, we can see that (although informally) there is a connection between the Kripke semantics we gave and the characteristic feature of the typed λ-calculus $\lambda^{\bigcirc}$. Further discussion can be found in Chapter 6.

3.5.3 Algebraic Semantics

In the classical setting, algebraic semantics and duality between frames and algebras are also studied well [41]. Intuitionistic analogues of this direction has been considered by Wolter and Zakharyaschev [44] in a general setting. Also, Jónsson-Tarski representation for constructive S4 and propositional lax logic have been considered by Alechina et al. [4].

Although we did not mention algebraic semantics for our LTL, it is not difficult to give a similar result. Consider a Heyting algebra equipped with a unary operation $\bigcirc$ preserving $\rightarrow$, and call such an algebra a $\bigcirc$-algebra. It is easy to see that the class of $\bigcirc$-algebras gives a semantics of our constructive LTL together with soundness and completeness. In a way similar to the classical case, we can establish Jónsson-Tarski representation for $\bigcirc$-algebras by defining translations between IM$^{\bigcirc}$-frames and $\bigcirc$-algebras.

3.5.4 Related Work

The natural deduction system introduced in Section 3.2 is similar to those for IMLs by Martini and Masini [25] and by Simpson [37] (aside from a few notational differences), which also use formulas with annotations indicating where the formulas hold. However, there are some differences between their systems and ours. First, while our $\bigcirc$I rule does not have any side condition, $\Box$-introduction rules by Martini and Masini requires that all time annotations in the antecedent must be smaller than $n+1$ (the time annotation of the succedent of the premise). There is also a similar condition in Simpson's one. Our $\bigcirc$I is actually more similar to $\Diamond$-introduction rule of Simpson's system. Second, $\vee$E and $\bot$E in our system are also different from theirs; these rules require time annotations of the succedents to be the same as the main formula, but it is not the case for Martini and Masini's and Simpson's systems. The absence of such a restriction allows us to prove the distributivity of $\Diamond$ over disjunction in their systems.

$\diamondsuit$ operator without distributivity or $\diamondsuit\bot \to \bot$ has also been discussed in the literature [42, 16, 21, 4]. In particular, Kripke semantics for propositional lax logic by Fairtlough and Mendler [16] and constructive S4 by Alechina et al. [4] had to consider *fallible worlds*, possible worlds at which any proposition becomes true. This direction will be treated in the Chapter 2.

Murphy et al. [27] consider a typed λ-calculus for distributed computation, which corresponds to intuitionistic S5 modal logic. Their system is based on natural deduction formalization by Simpson [37]. Although the system they formalized is an implicational fragment, they discuss how to add other connectives, and point out that $\bot$ and $\vee$ need special consideration. In particular, when $\vee$ is added, it is not obvious how to define operational semantics for case splitting. This is because, as they mention, the elimination rule for $\vee$ and $\bot$ in Simpson's system reasons *non-locally*, that is, main premise and conclusion may have different annotations. There is a similarity between this difficulty and our motivation not to admit distributivity, which is discussed in Section 3.1.

Since work by Davies and Pfenning [14] and Davies [13] on Curry-Howard correspondence for modal and temporal logic, many type systems for multi-stage languages based on their work have been proposed. Those languages typically include not only quasiquotation as in $\lambda^{\bigcirc}$ but also Lisp-like eval and lifting of values to code. As a result, their type systems could be seen as quite different modal logics: for example, the distributivity law would be validated if eval, which would have type $\bigcirc A \to A$, and lifting, which would have type $A \to \bigcirc A$, are supported in one language. The combination of these language features is motivated by a practical reason, rather than a correspondence with logics; it would also be interesting to investigate how these systems (more precisely, the corresponding logics) are characterized in terms of temporal or modal logics. $\lambda^{\bigcirc\Box}$ [45] and $\lambda^{\triangleright}$ [40] are examples of such investigations.

Chapter 4

Correspondence in Intuitionistic Modal Logic

4.1 Introduction

4.1.1 Background and Main Result

Modal logics describe various properties of relational structures such as reflexivity, transitivity, seriality, etc. These properties are often characterized by axioms of modal logics. For example, reflexivity is characterized by axiom **T** ($\Box p \to p$) in the sense that a structure is reflexive if and only if it validates this axiom. Similarly, transitivity and seriality are characterized by axioms **4** ($\Box p \to \Box\Box p$) and **D** ($\Box p \to \Diamond p$), respectively.

This relationship between properties and axioms is known as correspondence, and has been investigated in the area of (classical) modal logic [8]. Correspondence theory studies how modal axioms and properties of relational structures are related to each other. It describes an intuitive meaning of a modal axiom in terms of relational structures, and as a result we can know what kind of structure is implied by the modal axiom under consideration. Also, when we have some property of a relational structure we are interested in, from the correspondence theory we can know which axiom characterizes that property.

When reasoning about modal logics, we often find it helpful to move back and forth between syntactic and semantic entities (modal axioms and relational properties). An intuitionistic version of the correspondence theory will be useful in understanding and developing IMLs. Unfortunately, however, it seems that an intuitionistic version of the correspondence theory has not been extensively studied before.

This chapter considers correspondence in an intuitionistic setting. In particular, we consider which part of the existing classical correspondence results is also true in IML. More precisely, the problem we consider is as follows. First, consider an axiom X we are interested in. Then, on the one hand, we have a correspondent φ of X in classical Kripke frames. On the other hand, we have another correspondent ψ, in IM-frames (introduced in Definition 2.4.12). In this chapter we study when the two properties φ and ψ are the same.

For example, correspondence results for **T**, **4** and **D** mentioned above apply to IM-frames. However, for axiom **5** ($\Diamond p \to \Box\Diamond p$), it is not the case. This axiom classically corresponds to Euclidean property,[1] but intuitionistically it does not; there exists a non-Euclidean IM-frame that validates **5** (such an example will appear in Section 4.3).

We have two technical contributions in this chapter:

1. We introduce the notion of "robustness." This notion is defined in a fairly simple way, but still captures the "sameness" of the two correspondents (one for classical, and the other for intuitionistic).

2. We give a sufficient condition for robustness. We define a class of axioms in a syntactic way, and prove that all of its members are robust.

[1] A binary relation R is said to be Euclidean if it satisfies: if $x\ R\ y$ and $x\ R\ z$, then $y\ R\ z$.

Behind this result there is a classical result known as Sahlqvist's theorem. This theorem gives a sufficient condition for axioms to correspond to a first-order property [33]. The idea on which our result is based is similar to the idea found in this classical result.

4.1.2 Organization of This Chapter

Section 4.2 introduces several notations used in later sections. In the first part of Section 4.3, we discuss some examples of classical correspondence results in IM-frames. The second part introduces the notion of "robustness," with justification of its definition. In Section 4.4, we state and prove the main theorem, a sufficient condition for the robustness. Thanks to the simplicity of the definition of robustness, combined with the algebraic representation of Kripke semantics, the proof is fairly simple. As an example of possible application, in Section 4.5 we consider axioms arising in the application to security. We show some of them are robust, and using this fact we compare the strength of these axioms. Section 4.6 gives examples of robust and non-robust axioms. Section 4.7 summarizes this chapter, and makes some remarks, including related work.

4.2 Preliminaries

4.2.1 Kripke IM-frame and Semantics

In this chapter, we consider Kripke semantics using IM-frame. Since we discuss both classical and intuitionistic modal logics, it is convenient to define classical interpretation as well as the intuitionistic one (defined in the previous chapter) on the same frame structure.

Next we define classical and intuitionistic interpretations of modal formulas.

By an abuse of notation, we use IM-frame for both classical and intuitionistic interpretations. Usually the classical interpretation is defined on a pair $\langle W, R\rangle$ (i.e. a Kripke frame in the classical sense) rather than a triple, but ignoring the intuitionistic part $\leq$ we can regard IM-frames as Kripke frames.

We use $\mathcal{F}, V, w \Vdash^{cl} A$ to denote that A is classically true at world w in the frame $\mathcal{F}$ under the valuation V. For the intuitionistic interpretation, we use the symbol $\Vdash^{int}$ instead of $\Vdash^{cl}$. When $\mathcal{F}$ and V are clear from context, we simply write $w \Vdash^{cl} A$.

$\Vdash^{int}$ is already defined in Definition 2.4.4, and $\Vdash^{cl}$ is just the usual satisfaction relation of classical modal logic.

Definition 4.2.1. We write $\mathcal{F} \Vdash^{cl} A$ and say that A is *classically valid* in $\mathcal{F}$ (or $\mathcal{F}$ *classically validates* A) if for all V and w it holds that $\mathcal{F}, V, w \Vdash^{cl} A$. Intuitionistic version of validity $\mathcal{F} \Vdash^{int} A$ is defined in the same way.

Notation. For some notions introduced in this chapter, including the satisfaction and validity mentioned above, we have both classical and intuitionistic versions. In such a case, we use annotations *cl* and *int* to distinguish these versions.

The symbol $*$ is also used instead of *cl* or *int*. If $*$ appears in an equality, equivalence or other statement, it applies to both classical and intuitionistic versions.

4.2.2 Algebraic Representation

As usual, we can extend a valuation to the set of all formulas. Since there are two versions of semantics, we have two extensions of a valuation. We denote these extensions by V^{cl} and V^{int}. The precise definition is:

$$V^*(A) = \left\{ w \,\middle|\, \mathcal{F}, V, w \Vdash^* A \right\}.$$

Below we introduce auxiliary notations and list some properties of V^{cl} and V^{int} for later use.

Definition 4.2.2. 1. For each $X \subseteq W$, we write $k(X)$ for the greatest upward-closed subset of X. In other words, $k(X) = \{w \mid \leq[w] \subseteq X\}$.

2. For a valuation V, we define $V_\flat = k \circ V$. This is the greatest intuitionistic valuation less than or equal to V (with respect to the pointwise set-inclusion).

3. For each $X \subseteq W$, we define $l_R(X) = \{w \mid R[w] \subseteq X\}$. Also, we use $-$ for the set-theoretic complement, and $m_R(X)$ as a shorthand for $-l_R(-X)$.

It is easy to see that the following two hold for each valuation V: (1) $V_\flat$ is intuitionistic, and (2) $V_\flat = V$ if V is intuitionistic. So by abuse of notation we just write V^{int} to denote $(V_\flat)^{int}$ for non-intuitionistic valuation V.

It is easy to see that the following equalities hold for each valuation V.

$$
\begin{aligned}
V^*(A \wedge B) &= V^*(A) \cap V^*(B) & V^*(A \vee B) &= V^*(A) \cup V^*(B) \\
V^*(\Box A) &= l_R(V^*(A)) & V^{cl}(A \to B) &= \left(-V^{cl}(A)\right) \cup V^{cl}(B) \\
V^{cl}(\Diamond A) &= m_R(V^{cl}(A)) & V^{int}(p) &= V_\flat(p) \\
V^{int}(A \to B) &= k\left(\left(-V^{int}(A)\right) \cup V^{int}(B)\right) & V^{int}(\Diamond A) &= k(m_R(V^{int}(A)))
\end{aligned}
$$

Additionally, we would like to mention the two facts which play an important role in Section 4.4. First, a formula A is valid in $\mathcal{F}$ if and only if $V^*(A) = W$ for all valuations V on $\mathcal{F}$. This is clear from the definition. Second, the equality $l_R \circ k = l_R$ holds. This follows from the condition $(\leq ; R ; \leq) = R$ assumed in the definition of IM-frames.

4.3 Axioms and Correspondents

In this section we observe how well-known modal axioms can be characterized in terms of Kripke semantics. For some of them, their intuitionistic correspondents coincide with the classical ones, but for others they do not. After this observation, we discuss how this coincidence can be expressed formally, and introduce the notion of robustness.

4.3.1 Some Modal Axioms in IM-frames

Some of the well-known axioms intuitionistically correspond to exactly the same property as the classical one. Typical examples of such axioms are **T** and **4**, as mentioned in the introduction. For these two axioms, the classical method of deriving correspondents works perfectly in the intuitionistic case; take any possible world w and consider $V(p) = R(w)$. This attempt works because there exists such an *intuitionistic* valuation V.

Also, **D** ($\Box p \to \Diamond p$) and **C4** ($\Box\Box p \to \Box p$) have the same property, that is, their correspondents in the classical setting also apply to the intuitionistic case. **D** corresponds to seriality and **C4** corresponds to density. These correspondences can be verified in similar ways to the cases of **T** and **4**.

However, not all axioms necessarily correspond to the same property as the classical one. This can be observed in the work by Plotkin and Stirling in 1986 [32], although it is not mentioned explicitly. They considered a Kripke semantics (mentioned in Subsection 2.2.3), and gave a correspondent of Lemmon-Scott axiom schema $\Diamond^k \Box^l p \to \Box^m \Diamond^n p$. According to their result, to express the correspondent of this schema we need to use both R and $\leq$. Therefore we can see that (most of) the instances of Lemmon-Scott schema intuitionistically correspond to different properties from the classical case.

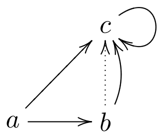
Although their result does not directly apply to our setting (because we are considering a different semantics from theirs), the situation is similar. Indeed, it is not difficult to find an axiom whose classical and intuitionistic correspondents are different. Some of them are listed in Table 4.1. All of these are instances of Lemmon-Scott axiom schema. In Figure 4.1 we list examples of IM-frames which intuitionistically validate these axioms but classically do not (the specified properties in Table 4.1 do not hold at a of each frame). In this figure, solid and dotted arrows represent accessibility relations R and $\leq$, respectively.

4.3.2 Defining Robustness

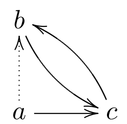
The series of examples above motivates the following question: which axiom corresponds to the same property in both classical and intuitionistic modal logics? This question will be discussed in the next section. In this subsection we are going to make a preparatory discussion on how to formalize the "sameness of the classical and intuitionistic correspondents."

axiom		classical correspondent
5	$\Diamond p \to \Box\Diamond p$	Euclidean (if $x\ R\ y$ and $x\ R\ z$, then $y\ R\ z$)
B	$p \to \Box\Diamond p$	symmetric (if $x\ R\ y$, then $y\ R\ x$)
$\mathbf{T}_\Diamond$	$p \to \Diamond p$	reflexive ($x\ R\ x$)
$\mathbf{4}_\Diamond$	$\Diamond\Diamond p \to \Diamond p$	transitive (if $x\ R\ y$ and $y\ R\ z$, then $x\ R\ z$)
CD	$\Diamond p \to \Box p$	unique (if $x\ R\ y$ and $x\ R\ z$, then $y = z$)
C	$\Diamond\Box p \to \Box\Diamond p$	confluent (if $x\ R\ y$ and $x\ R\ z$, then $\exists u.\ (y\ R\ u$ and $z\ R\ u))$

Table 4.1: Examples of axioms whose classical and intuitionistic correspondent are not the same.

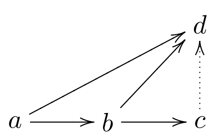


IM-frame for **5** which is not Euclidean

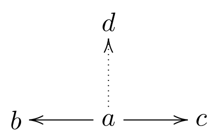


IM-frame for **B** which is not symmetric

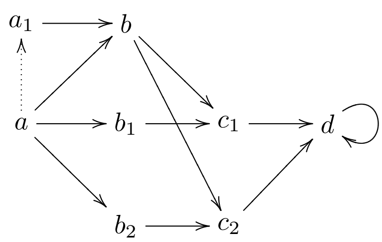
IM-frame for $\mathbf{T}_\Diamond$ with irreflexive point



IM-frame for $\mathbf{4}_\Diamond$ which is not transitive



IM-frame for **CD** in which successor is not unique



IM-frame for **C** without confluence

Figure 4.1: Examples of IM-frames which intuitionistically validate axioms, but not classically

Let A be an axiom, and assume that A classically corresponds to a property φ, and intuitionistically corresponds to ψ.[2] If we write $\mathcal{F} \models \chi$ to mean that $\mathcal{F}$ satisfies a property χ, this assumption means that

$$\mathcal{F} \Vdash^{cl} A \iff \mathcal{F} \models \varphi, \text{ and } \mathcal{F} \Vdash^{int} A \iff \mathcal{F} \models \psi$$

for any IM-frame $\mathcal{F}$. In this setting, what does it mean for frame conditions φ and ψ to be the same? Here φ and ψ are properties of IM-frames, so it is natural to regard them as the same if and only if they are satisfied by exactly the same IM-frames. That is, φ and ψ are "the same" if and only if

$$\mathcal{F} \models \varphi \iff \mathcal{F} \models \psi$$

holds for any IM-frame $\mathcal{F}$.

From the three equivalences above, we can see that the classical and intuitionistic correspondents for A are the same if and only if

$$\mathcal{F} \Vdash^{cl} A \iff \mathcal{F} \Vdash^{int} A$$

is the case for any $\mathcal{F}$.

Now we have arrived at the following definition.

Definition 4.3.1. Let A be a formula.

1. A is said to be *CI-stable* if, for any IM-frame $\mathcal{F}$, if $\mathcal{F} \Vdash^{cl} A$ then $\mathcal{F} \Vdash^{int} A$.
2. A is said to be *IC-stable* if, for any IM-frame $\mathcal{F}$, if $\mathcal{F} \Vdash^{int} A$ then $\mathcal{F} \Vdash^{cl} A$.
3. A is said to be *robust* if it is both CI-stable and IC-stable.

The intended meanings of these notions are explained as follows. Suppose that a formula A classically corresponds to a certain property φ. Then, CI-stability of A means that if an IM-frame $\mathcal{F}$ satisfies φ, then A is intuitionistically valid in $\mathcal{F}$. On the other hand, IC-stability of A means that if an IM-frame $\mathcal{F}$ *does not* satisfy φ, then A is *not* intuitionistically valid in $\mathcal{F}$. Accordingly, robustness of A means that A has the same correspondent φ in classical and intuitionistic modal logics, as discussed above.

Example 4.3.2. *1. Axioms* **T***,* **4***,* **D***, and* **C4** *are all robust.*

2. **5***,* **B***,* $\mathbf{T}_\diamond$*,* $\mathbf{4}_\diamond$*,* **CD***, and* **C** *are not robust. These axioms are all CI-stable, but not IC-stable.*

Robustness of axioms in 1 follows from Theorem 4.4.4, and CI-stability in 2 follows from Proposition 4.4.6, clause 1.

Incidentally, the robustness is also characterized in terms of correspondents as follows.

Proposition 4.3.3. *A formula A is robust if and only if its intuitionistic correspondent for A can be expressed without using $\leq$.*

Proof. If A is robust, then its intuitionistic correspondent is the same as the classical one, so "if" part is clear. We prove the other direction. For an IM-frame $\mathcal{F} = \langle W, \leq, R \rangle$, define $\mathcal{F}^* := \langle W, \Delta, R \rangle$, where $\Delta = \{(x, x) \mid x \in W\}$ is the identity relation. Since Δ is an ordering and $(\Delta\,;R\,;\Delta) = R$, this defines another IM-frame. It is easy to see that

1. if a property φ can be expressed without $\leq$, then $\mathcal{F} \models \varphi \iff \mathcal{F}^* \models \varphi$, and
2. $\mathcal{F} \Vdash^{cl} A \iff \mathcal{F}^* \Vdash^{int} A$.

The assertion follows from these two facts. $\square$

4.4 A Sufficient Condition for Robustness

In this section, we give a sufficient condition for robustness. First we describe the main idea by considering a simple case, and next we extend this result to a more general form. There is an analogy between our result and a classical result by Sahlqvist [33], and the proof presented below is partly based on the algebraic proof of Sahlqvist's theorem [34, 8].

[2] Formally, φ and ψ can be written as (second-order) sentences in an appropriate signature. The argument below can be written more formally by using this representation, but here we do not need the details.

4.4.1 Basic Idea

Sahlqvist showed that an implication $A \to B$ corresponds to a first-order property, if B is positive and A satisfies a certain condition. This condition mentions positions of occurrences of $\to$ and $\Box$; it restricts places in which these two connectives can occur.

Our result on robustness states, similarly to Sahlqvist's theorem, an implication $A \to B$ is robust if A and B are positive and satisfy some conditions. These conditions are, again similarly to Sahlqvist's one, phrased as conditions on occurrences of $\to$, $\Box$, and $\Diamond$.

Here we consider axioms of the form $A \to B$ only. For an axiom of this form, we have $\mathcal{F} \Vdash^* A \to B$ if and only if

$$\forall V, w.(\mathcal{F}, V, w \Vdash^* A \implies \mathcal{F}, V, w \Vdash^* B)$$

holds. Using the notation V^*, we can identify the validity of implication with the set inclusion, as shown in the following lemma.

Lemma 4.4.1. *Let A and B be formulas. Then $\mathcal{F} \Vdash^* A \to B$ if and only if $V^*(A) \subseteq V^*(B)$ for any valuation V on $\mathcal{F}$.*

So the robustness of $A \to B$ is equivalent to: $V^{cl}(A) \subseteq V^{cl}(B)$ for all V, if and only if $V^{int}(A) \subseteq V^{int}(B)$ for all V. This representation of robustness is convenient for our purpose, and it is the reason why we introduced algebraic representation.

To sketch the main idea, here we consider the simplest case. If we have $V^{cl}(A) = V^{int}(A)$ and $V^{cl}(B) = V^{int}(B)$ for every V, then $A \to B$ is clearly robust. When do these equalities hold? If we attempt to prove $V^{cl}(A) = V^{int}(A)$ by induction on A, obviously there are problems in three cases: p, $\Diamond$, and $\to$. In these cases V^{cl} and V^{int} do not agree, because k is applied in clauses for V^{int}.

Here is a key observation: l_R absorbs k, that is, $l_R \circ k = l_R$ holds, as mentioned at the end of Section 4.2. So, although in general we do not have

$$V^{cl}(A) = V^{int}(A) \implies V^{cl}(\Diamond A) = V^{int}(\Diamond A),$$

instead we can say that

$$V^{cl}(A) = V^{int}(A) \implies V^{cl}(\Box\Diamond A) = V^{int}(\Box\Diamond A)$$

holds, and similarly for p and $\to$. This fact motivates the following definition.

Definition 4.4.2. An occurrence of an atom or a connective in a formula is said to be *protected* if it is immediately preceded by $\Box$.

For example, in $\Box p \wedge \Box\Diamond q$, the occurrences of p and $\Diamond$ are protected, but q is not.

From the argument above, we can conclude that if all occurrences of atoms, $\Diamond$, and $\to$ in A and B are protected, then $A \to B$ is robust. **4** is an example of such an axiom. In the next subsection, we are going to discuss how far this condition can be relaxed.

4.4.2 A Class of Robust Axioms

Definition 4.4.3. A formula A is said to be

1. *positive* if A does not contain $\to$;
2. *$\Diamond$-protected* if all occurrences of $\Diamond$ in A are protected;
3. *atom-protected* if all occurrences of atoms, $\bot$ and $\top$ excepted, in A are protected;
4. *protected* if A is both $\Diamond$-protected and atom-protected.

Using these terminologies, we can state the main theorem as follows.

Theorem 4.4.4. *Let A be a protected positive formula, and B a $\Diamond$-protected positive formula. Then, $A \to B$ and $A \to \Diamond B$ are robust.*

Below we are going to prove this theorem.

Lemma 4.4.5. *Let A be a formula and V a valuation.*

1. *If A is positive, then $V^{int}(A) \subseteq {V_\flat}^{cl}(A) \subseteq V^{cl}(A)$.*[3]
2. *If A is positive and $\Diamond$-protected, then $V^{int}(A) = {V_\flat}^{cl}(A)$ and $V^{int}(\Diamond A) = k({V_\flat}^{cl}(\Diamond A))$.*
3. *If A is atom-protected, then ${V_\flat}^{cl}(A) = V^{cl}(A)$.*

Proof. By induction on the construction of A, using the following facts:

1. positive formulas are constructed from atoms by $\wedge$, $\vee$, $\Box$, and $\Diamond$;
2. positive $\Diamond$-protected formulas are constructed from atoms by $\wedge$, $\vee$, $\Box$, and $\Box\Diamond$;
3. atom-protected formulas are constructed from protected atoms, $\top$ and $\bot$ by freely applying any connectives.

In induction steps for 2, we use the equality $l_R \circ k = l_R$. $\Box$

Proposition 4.4.6. *Let A and B be positive formulas.*

1. *If B is $\Diamond$-protected, then $A \to B$ and $A \to \Diamond B$ are CI-stable.*
2. *If A is protected, then $A \to B$ is IC-stable.*

Proof. Check that

$$V^{int}(A) \subseteq {V_\flat}^{cl}(A) \subseteq {V_\flat}^{cl}(B) = V^{int}(B)$$

and

$$V^{int}(A) \subseteq k\left({V_\flat}^{cl}(A)\right) \subseteq k\left({V_\flat}^{cl}(\Diamond B)\right) = V^{int}(\Diamond B)$$

for 1, and

$$V^{cl}(A) = V^{int}(A) \subseteq V^{int}(B) \subseteq V^{cl}(B)$$

for 2, using Lemma 4.4.1. $\Box$

From this proposition, it is clear that Theorem 4.4.4 holds. In the same way as Sahlqvist's theorem, we can extend this theorem.

Proposition 4.4.7. *1. If A and B are robust, then so is $A \wedge B$.*

2. If A and B are robust and they do not share any atom, then $A \vee B$ is also robust.

3. If A is robust, then so is $\Box A$.

Proof. 1 and 2 are easy. For 3, take any IM-frame $\mathcal{F}$ and let $\mathcal{F}'$ be the subframe of $\mathcal{F}$ generated by all R-successors (that is, all $w \in W$ such that for some $w' \in W$ it holds that $w' \mathrel{R} w$). Then we can prove that $\mathcal{F} \Vdash^* \Box A$ if and only if $\mathcal{F}' \Vdash^* A$, from which robustness of $\Box A$ follows. $\Box$

Remark 4.4.8. Above we assumed that A and B are positive. Actually, in some ways this assumption can be relaxed a little. For example, if all occurrences of atoms, $\Diamond$, and $\to$ in A are protected, then the same result holds for non-positive A, because for such A it holds that $V^{cl}(A) = V^{int}(A)$, as discussed in the previous subsection. There would be some similar ways to extend our result.

However, this looks ad-hoc and makes things complicated, so here we do not investigate them further.

4.5 Axioms from Access Control Logics

In this section, we are going to consider some of the modal axioms appearing in security, and compare their strength using their correspondents. Although similar results have already been mentioned in recent literature [2, 9], the approach presented here is more systematic. This would be an example showing that a general theory on correspondence can be a useful foundation for comparing various modal axioms.

[3] ${V_\flat}^{cl}$ stands for $(V_\flat)^{cl}$, not $(V^{cl})_\flat$.

4.5.1 Modality and Access Control

In the context of access control, modal logic has been considered as a basic framework to express access control policies, which determine whether a principal (user, program, machine, or other entity) may access a resource. Logics which express a policy as a formula is called access control logic, and have been studied recently.

In access control logics, a special operator, called `says` operator, plays a central role in expressing policies. This operator is used in the form A `says` s, where A is a principal and s is a formula, and the whole expression is again a formula. Intuitively this formula means that "A supports the statement s" (although there seem to be some variations in exact interpretation of the `says` operator). Regarding A `says` as a modality indexed by A, we can formalize an access control logic as a kind of multimodal logic.

As a logic for representing policies, access control logics usually require additional axioms concerning `says` operator other than necessitation and normality. For example, one may want to assume (`admin says` $s) \to s$ as an additional axiom on `says` operator. When s represents an operation of deleting or modifying files, this axiom allows administrators to delete or modify files when they request.

Since different access control logics may interpret the `says` operator differently, there are several options in axiomatization of access control logics. To investigate room for choice, Abadi studied the consequence relation between some axioms in classical and intuitionistic modal logics [2]. His result implies that some intuitively natural principles derive an undesirable axiom (in particular, in classical setting) which make the logic degenerate. He also proved that some axioms do not prove such an axiom in intuitionistic setting, using proof-theoretic technique. What we are going to do is to obtain similar results more systematically, using results we have seen in the previous section.

4.5.2 Axioms and their Robustness

In what follows, we will write $\Box_A$ instead of A `says` , and suppress the principal annotation A when it is not significant. We consider **C4** having appeared before and the following axioms previously considered in relation to access control logics [2, 17, 9].

- **Unit**: $p \to \Box p$
- **Bind**: $(p \to \Box q) \to (\Box p \to \Box q)$
- **Hand-off**: $\Box_A(B \Rightarrow A) \to (B \Rightarrow A)$
- **Escalation**: $\Box p \to (p \vee \Box\bot)$

It is easy to see that, under the presence of **Unit**, axioms **Bind** and **C4** are (intuitionistically) equivalent. These three axioms come from lax logic, which is a version of intuitionistic modal logic [16]. Lax logic is known to be a logical foundation of computational lambda calculus [26, 21, 7], and recently a calculus for access control, called CDD [1], has been proposed as a variant of computational lambda calculus.

In **Hand-off** axiom, a new connective $\Rightarrow$ appears. The formula $B \Rightarrow A$ means that "B speaks for A," and $\Rightarrow$ is called "speaks for" operator. If we allow universal quantification over formulas, $\Rightarrow$ is defined by

$$B \Rightarrow A \equiv \forall p.((\Box_B p) \to (\Box_A p)).$$

Hand-off axiom states that if a principal A says that some principal B speaks for A, then B actually speaks for A. This axiom formalizes the delegation of authority.

Escalation is considered as an undesirable axiom, since a modality satisfying this axiom is considered degenerate; if we read $\Box$ as "says," **Escalation** on $\Box_A$ implies that either everything a principal A says is true, or A says anything [2].

Below we are going to compare the strength of these axioms, based on semantic method using robustness result. In particular, we consider which axiom does not imply which.

Before doing that, we adjust the definition of Kripke semantics so that it can interpret $\Box_A$ and $\Rightarrow$. To interpret $\Box_A$, instead of a single accessibility relation R, we employ a family of relations R_A indexed by principals. Each R_A is assumed to satisfy the same constraint as R of IM-frames. Then, $\Box_A$ is naturally interpreted, as the necessity with respect to R_A in the new setting. We regard $B \Rightarrow A$ as a propositional constant, and thus it is interpreted as a fixed set of possible worlds. We define its classical interpretation by

$$[\![B \Rightarrow A]\!] := \{x \mid R_A\,[x] \subseteq R_B\,[x]\}\,.$$

In other words, $B \Rightarrow A$ is true at x if and only if $x\ R_A\ y$ implies $x\ R_B\ y$ for all y. This is consistent with the "definition" $B \Rightarrow A \equiv \forall p.((\Box_B p) \to (\Box_A p))$. Intuitionistic interpretation is just $k([\![B \Rightarrow A]\!])$.

Proposition 4.5.1. ***Escalation**, **C4** and **Hand-off** are robust, whereas **Unit** and **Bind** are not.*

Proof. The robustness of these three axioms follows from Theorem 4.4.4 (note that $B \Rightarrow A$ can be treated as an atom).

Non-robustness of **Unit** and **Bind** are easily seen from the next proposition, in which we will see that the classical and intuitionistic correspondents for these axioms are distinct. $\Box$

4.5.3 Comparing Strength of Axioms

It is not difficult to check the following correspondences.

Proposition 4.5.2.
1. ***C4** corresponds to: $R \subseteq R^2$.*
2. ***Unit** classically corresponds to $R \subseteq \Delta$, where Δ is a diagonal relation, and intuitionistically corresponds to $R \subseteq\ \leq$.*
3. *Classically **Bind** is equivalent to **Escalation**, and therefore they have the same correspondence in classical setting. Intuitionistically **Bind** corresponds to:*

$$R\,[x] \subset R\,[\leq [x] \cap R\,[x]]\,.$$

4. ***Escalation** corresponds to: $R\,[x] \neq \emptyset \implies x\ R\ x$.*
5. ***Hand-off** corresponds to the following condition:*

$$(\forall y \in R_A\,[x]\,.(R_A\,[y] \subseteq R_B\,[y])) \implies R_A\,[x] \subseteq R_B\,[x]\,.$$

Since **Escalation**, **C4**, and **Hand-off** are robust, we do not distinguish classical and intuitionistic setting when considering their correspondents. Similar results in a slightly different setting are also mentioned by Boella, Gabbay, Genovese and Torre [9].

From these results, the non-derivability of some axioms from other axioms. For example:

- **C4** does not imply **Bind**, and **Bind** does not imply **Escalation**.
- **Hand-off** does not imply **C4** nor **Unit**. This is easily checked from the fact that a frame with $R_A = R_B$ for any principals A and B admits **Hand-off**, but not necessarily **C4**, nor **Unit**.
- **Escalation** does not imply **Unit**. This is also an easy consequence of the correspondence result. If we consider the following frame, it satisfies **Escalation** but not **Unit**, since $R\,[a] = \{a, b\} \not\subseteq \{a\}$.

$$\begin{tikzcd} a \arrow[loop left] \arrow[r] & b \end{tikzcd}$$

- **Unit** does not imply **C4** nor **Hand-off**. To see this, consider the following frame.

$$\begin{tikzcd} b \\ a \arrow[u, dotted] \arrow[u, bend right] \end{tikzcd}$$

  This satisfies **Unit**, but not **C4**. For **Hand-off**, regard the solid arrow as R_A and let $R_B = \emptyset$. Then this frame satisfies **Unit**, but it falsifies **Hand-off**.

4.6 Some More Examples

In this section, we are going to discuss some more examples of modal axioms.

4.6.1 Limitation of Our Method

The class of formulas covered by Theorem 4.4.4 and Proposition 4.4.7 contains all robust formulas listed in Section 4.3. However, there exists a robust formula outside the scope of the theorem, even if we consider axioms of the form $A \to B$ for positive A, B only. For example, let $A = \Box p$ and $B = p \vee \Diamond p$, and consider the axiom $\mathbf{D'} = A \to B$. Theorem 4.4.4 does not apply to $\mathbf{D'}$ since p on the right-hand side of $\to$ is not protected. However, this axiom is robust, because $\mathbf{D'}$ is valid in $\langle W, \leq, R\rangle$ if and only if R is serial (in both classical and intuitionistic interpretations).

For this axiom, our method used above does not seem to work. This is because, if $A \to B$ can be proved to be robust in our method, it actually means a little more than robustness; for such A and B, it holds that ${V_\flat}^{cl}(A) \subseteq {V_\flat}^{cl}(B) \iff V^{int}(A) \subseteq V^{int}(B)$. However, for the choice of A and B above, this equivalence does not hold. Indeed, take the frame

$$\begin{tikzcd} b & \\ a \arrow[u, dotted] \arrow[r] & c \end{tikzcd}$$

and consider $V(p) = \{b, c\}$. Then, after a little calculation we obtain $V^{cl}(A) = V^{int}(A) = V^{cl}(B) = \{a, b, c\}$ but $V^{int}(B) = \{b, c\}$.

$\mathbf{D'}$ is not an axiom motivated from some meaningful application. It is constructed as a counterexample, and does not seem a natural axiom since p on the right-hand side is redundant. There could be more meaningful counterexample, but at the time of writing we do not have such an example.

4.6.2 An IC-Stable, but not CI-stable Axiom

The examples of non-robust axioms that appeared above are all CI-stable, but not IC-stable. It is a natural question whether there exists an axiom which is IC-stable, but not CI-stable.

Perhaps the simplest example is $\Box p \to \Diamond\Diamond p$. It is easy to check that this axiom is classically valid in a frame $\langle W, R\rangle$ if and only if $R \cap R^2$ is serial (in other words, R has the property that for all $w \in W$ there exists $v \in W$ such that both $w\ R\ v$ and $w\ R^2\ v$ holds). On the other hand, there exists an IM-frame in which $R \cap R^2$ is serial, but the said axiom is not valid. For example, consider the frame

$$\begin{tikzcd} & c \arrow[r] & d \arrow[loop right] \\ a \arrow[ur] \arrow[r] & b \arrow[u, dotted] \arrow[u, bend right] \arrow[ur] \arrow[loop right] & \end{tikzcd}$$

and a valuation $V(p) = \{b, c\}$. Then we have $a \Vdash^{int} \Box p$, but $a \nVdash^{int} \Diamond\Diamond p$. To check the latter, note that c has no successor satisfying p. So $c \nVdash^{int} \Diamond p$, and from heredity we also have $b \nVdash^{int} \Diamond p$. This means that a has no successor satisfying $\Diamond p$, so a does not satisfy $\Diamond\Diamond p$.

4.6.3 Löb axiom

Above we considered axioms of the form $A \to B$ where A and B are positive. As an example of robust formulas which does not have this form, we consider Löb axiom (**GL**), which takes the form $\Box(\Box p \to p) \to \Box p$. In classical modal logic, this axiom is known to correspond to the conjunction of transitivity and non-existence of infinite path [8]. Let us take a look at the sketch of the proof.

For transitivity, assume $u\ R\ v\ R\ w$ and let V be a valuation such that $x \in V(p)$ if and only if neither $x = v$ nor $x = w$ holds. Then, from the contrapositive of **GL** there exists some R-successor of u at which $\Box p \to p$ does not hold. On the other hand we can show that, at any world other than w we have $\Box p \to p$. Therefore $u\ R\ w$ needs to hold.

For non-existence of infinite path, define V by: $w \in V(p)$ if and only if there is no infinite path starting from w. Then under this valuation $\Box p \to p$ is true everywhere, so applying **GL** we obtain $V(p) = W$. This means that there is no infinite path.

This proof is done in the classical setting, but almost the same method works for IM-frame, too. We only need to fix the definition of V in the first part. Instead of "neither $x = v$ nor $x = w$" we need to consider "neither $x \leq v$ nor $x \leq w$" because otherwise V may fail to be an intuitionistic valuation.

Interestingly, we can also use algebraic argument to prove the same result. A little calculation shows that **GL** is classically valid if and only if all valuations V satisfy

$$l_R((-l_R(V(p))) \cup V(p)) \subseteq l_R(V(p)),$$

and intuitionistically

$$l_R((-l_R(V(p))) \cup k(V(p))) \subseteq l_R(V(p)).$$

Therefore, to see **GL** is robust it is sufficient to show that

$$k(X \cup Y) = k(X \cup k(Y)),$$

where $X = -l_R(V(p))$ and $Y = V(p)$. The right-to-left inclusion is obvious. For the converse, take $x \in k(X \cup Y)$, and let U be the set of all upper bounds of x. Then we have $U \subseteq X \cup Y$, hence $U \setminus X \subseteq Y$. Actually we can say $U \setminus X \subseteq k(Y)$, because the left-hand side is the intersection of the two upward-closed sets, namely U and $-X = l_R(V(p))$. Therefore we have

$$U = (U \cap X) \cup (U \setminus X) \subseteq X \cup k(Y),$$

and hence $x \in k(X \cup k(Y))$.

4.6.4 Grzegorczyk axiom

As another famous example of an axiom of a complicated form, we have Grzegorczyk axiom (**Grz**) $\Box(\Box(p \to \Box p) \to p) \to p$. Classically, **Grz** is known to correspond to reflexive and transitive frames in which there is no nontrivial infinite path [8] (by "nontrivial" infinite path we mean a path $x_1 \; R \; x_2 \; R \cdots$ such that $x_n \neq x_{n+1}$ for any n). This condition resembles that for **GL**, but unlike **GL**, **Grz** is not robust.

It holds that if an IM-frame $\mathcal{F}$ is reflexive, transitive, and has no nontrivial infinite R-path, then $\mathcal{F}$ validates **Grz**. However, the converse does not hold; there exists an IM-frame with nontrivial infinite R-path, in which **Grz** is valid. For example, let W be the set of all natural numbers and both $\leq$ and R be the usual ordering on natural numbers. Then we can easily check that, from heredity, p and $\Box p$ are equivalent at each world in this frame. So in this frame **Grz** is valid, although it contains nontrivial infinite path $(1, 2, 3, \cdots$, for example).

Actually, in an IM-frame, **Grz** corresponds to the conjunction of reflexivity, transitivity and the following condition: there exists no pair of an upward-closed (with respect to $\leq$) set X and an infinite path $x_1 \; R \; x_2 \; R \cdots$, such that $x_n \in X$ if and only if n is even. By transitivity, this means that **Grz** is valid if and only if there is no infinite path on which truth of p can alter infinitely many times.

4.7 Summary and Remarks

4.7.1 Summary

We studied correspondence of axioms and properties of frames in IM-frames, and there are two main observations. First, the sameness of classical and intuitionistic correspondents can be captured by the equivalence in classical and intuitionistic semantics. Second, there exists a syntactically defined class of axioms, for whose members the correspondents in the classical and intuitionistic settings are the same.

The semantics considered in this work is the one defined by Wolter and Zakharyaschev [44]. It seems that the condition $(\leq; R; \leq) = R$ plays an important role. In particular, we used the equality $l_R \circ k = l_R$ in the proof of Lemma 4.4.5, which is a consequence of this condition.

To give a class of robust axioms syntactically, the main strategy we took is to restrict occurrences of "problematic" constructs (atoms, $\to$ and $\Diamond$), which require intuitionistic counterpart $\leq$ to interpret. A similar method is used in Sahlqvist's theorem, in which mainly $\to$ and $\Box$ are restricted.

At first sight the class we gave may seem to be small because it strongly restricts occurrences of an atom and $\Diamond$. However, it contains most part of standard robust axioms. At the time of writing, we could find only a few examples of robust formulas outside this class. To make significant improvement in the current result, some new idea would be required.

Also, as a possible application, we have considered modal axioms from security. Using model theoretic argument together with robustness results, we have compared strength of these axioms. This demonstrates that the result presented in this chapter can be a tool for investigating modal logic in the intuitionistic setting.

4.7.2 Similar Result for Other Semantics

Our argument above is based on IM-model, and its technical details depend on the choice of semantics. Here we briefly discuss how to develop a similar result for other semantics.

Let us consider IR-model instead of IM-model, where we do not have any constraint on $\leq$ and R, and the truth condition of $\Box A$ reads

$$x \Vdash^{int} \Box A \iff \text{if } x \leq x' \text{ and } x' \, R \, x'', \text{ then } x'' \Vdash^{int} A.$$

In this setting, we need to use $\leq$ as well as R to express correspondents concerning $\Box$ (the situation is the same as the problem we found in Section 4.4, where occurrences of $\Diamond$ caused a problem). As a result, it seems that most part of the classical correspondence results are not true for this semantics; even **T** and **4** are excluded. So, if the notion of robustness is left unchanged, the same problem for this semantics would not be interesting.

One possible approach in such a setting is to use the result on IM-frames indirectly, by giving a translation from non-IM-frames to IM-frames. If we do not consider $\Diamond$, or we admit duality $\Diamond p \equiv \neg\Box\neg p$ (which is a theorem in IM-frame semantics), it is not difficult to translate the semantics defined on non-IM-frame into IM-frame semantics. This is because, from heredity, we have $x \Vdash^{int} \Box A$ in this semantics if and only if "if $x(\leq ; R ; \leq)x'$, then $x' \Vdash^{int} A$" holds. Therefore what we actually need to focus on is not R, but the composite $R' = (\leq ; R ; \leq)$. Regarding R' as new R, we obtain IM-frame semantics. Therefore, if an axiom X corresponds to some property $\varphi(R)$ in IM-frame semantics, then the same axiom corresponds in non-IM-frame semantics to $\varphi(\leq ; R ; \leq)$. So we can say that if X is robust, then its intuitionistic correspondent in non-IM-frames can be obtained by replacing all occurrences of R in the classical correspondent with $(\leq ; R ; \leq)$.

4.7.3 Local Version of Robustness

In Section 4.3 we have defined robustness of an axiom A by

$$\forall \mathcal{F}.(\mathcal{F} \Vdash^{cl} A \iff \mathcal{F} \Vdash^{int} A).$$

This is a global notion in the sense that it concerns global validity only. We could also consider its local version like

$$\forall \mathcal{F}, x.(x \Vdash^{cl} A \iff x \Vdash^{int} A)$$

(x ranges over the set of possible worlds of $\mathcal{F}$), which seems to be a natural definition of local robustness. However, axioms satisfying this condition would be rare. This is because the intuitionistic interpretation has heredity, while the classical one does not. For example, consider an IM-frame consisting of two comparable points $a \leq b$, and assume a is reflexive and b is not ($R = \{(a,a),(a,b)\}$). Then, classically **T**, **4** and **D** are locally valid at a, but intuitionistically they are not.

One possibility to adjust the definition of local robustness is to consider the following condition:

$$\forall \mathcal{F}, x.((\forall x'.x \leq x' \implies x' \Vdash^{cl} A) \iff x \Vdash^{int} A).$$

If we define local robustness of A by this condition, then the same result as Theorem 4.4.4 is proved by a similar argument.

4.7.4 Related Work

It seems that the problem considered in this chapter has not been studied before. However, there are a few studies concerning correspondence for concrete modal axioms in intuitionistic settings.

One of such studies has been done by Sotirov [38]. He introduced two modal accessibilities R and R^* independently to interpret $\Box$ and $\Diamond$, and gave corresponding properties for various axioms in terms of three accessibility relations $\leq$, R and R^*. These axioms include **T**, $\mathbf{T}_\Diamond$, **4**, $\mathbf{4}_\Diamond$, **B**, and **D**.

In his result, because of the existence of independent R^*, occurrences of $\Diamond$ do not introduce extra $\leq$ in expressing the corresponding properties. As a result, in contrast with ours, it would be the case that the occurrences of $\Diamond$ does not interfere robustness (but an appropriate definition of robustness in his setting is not obvious because of the independence of R and R^*).

He also considered a lot of axioms containing nested $\rightarrow$ (and $\neg$), which we did not handle. In his setting, as well as ours, nested implications cause non-robustness of axioms (that is, we need to mention $\leq$ to express the correspondents). For example, $\Box A \rightarrow \neg\neg A$, which classically corresponds to reflexivity, corresponds to the following property: if $x \leq y$, then there exists z such that $y \leq z$ and $x \; R \; z$.

The work by Plotkin and Stirling [32] mentioned in Section 4.3 is another example. They defined a birelational Kripke semantics, discussed some concrete axioms, and identified their correspondents.

According to their result, the Lemmon-Scott axiom schema $\Diamond^k \Box^l p \rightarrow \Box^m \Diamond^n p$ corresponds to the following property: if $w \; R^k \; u$ and $w \; R^m \; v$, then there exists u' and x such that $u \leq u'$, $u' \; R^l \; x$, and $v \; R^n \; x$. From this result we can see that, in their setting, as well as ours, many axioms would correspond to properties different from the classical case.

They also considered the axiom $\Diamond p \wedge \Diamond q \rightarrow \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p)$, which, when added to S4, results in S4.3. The correspondent of this axiom they gave is also different from the classical one previously known (if $x \; R \; y$ and $x \; R \; z$, then either $y \; R \; z$ or $z \; R \; y$ holds).

Also, it would be worth noting that they pointed out that it is unlikely that classically equivalent axioms ($\Box p \rightarrow p$ and $p \rightarrow \Diamond p$, for example) always correspond to the same property. Their result shows that this is indeed the case.

Chapter 5

Neighborhood Semantics

5.1 Introduction

5.1.1 Background and Motivation

Semantics for IML has been considered mainly in Kripke-style, but there is a difficulty in such an approach: the usual interpretation of $\Diamond$ (and $\vee$) validates the distributivity. This is roughly because existential quantification, which is used to interpret $\Diamond$ in the meta-level, distributes over disjunction. So, to avoid the distributivity, we need to fix the interpretation of $\Diamond$ [42, 4]. As a result of such a modification, the analogy between intuitionistic and classical modal logic is lost; this makes IMLs harder to understand as a variant of classical modal logics. For example, as we have seen in Chapter 4, the correspondence between axioms and properties of frames can be lost.

In the classical setting, it is known that a modality $\Diamond$ without distributivity cannot be handled in the usual Kripke semantics, and such modalities is said to be *non-normal*. One of the alternative tools to study such a modality is neighborhood semantics [12, 29]. However, its intuitionistic version has not been extensively studied. Although Sotirov [38] and Wijesekera [42] considered neighborhood semantics for certain IMLs, it does not seem that they tried to capture the nature of non-distributive $\Diamond$ in terms of neighborhood semantics. In Sotirov's work only a necessity modality is considered, and Wijesekera's semantics requires some extra axioms for completeness.

In this chapter, we will investigate IMLs without distributivity as a *non-normal* modal logic. This point of view has not been considered before; in preceding researches, $\Diamond$ without distributivity in intuitionistic setting has not been referred to as non-normal. We will demonstrate that actually it is natural to consider them as non-normal modal logics, with intuitionistic base logic. This is achieved by showing that the neighborhood semantics, which has been developed to capture classical non-normal modal logics, is naturally extended to the intuitionistic setting.

5.1.2 Overview

In this chapter, we will consider neighborhood semantics (which differs from Sotirov's or Wijesekera's ones) for IML. We will discuss

1. the relationship between relational semantics defined in Chapter 2 and our neighborhood semantics, and

2. the classical case of our framework, and the relationship with classical monotone and bimodal logics.

Neighborhood semantics for IML is basically obtained by adding a preorder (taken from ordinary Kripke semantics for intuitionistic logic) to the classical neighborhood semantics. So a neighborhood frame is a triple $\langle W, \leq, N\rangle$, where N is a neighborhood function, a mapping from W to $\mathcal{P}(\mathcal{P}(W))$.

For subject (1), we show that relational and neighborhood semantics are "almost" equivalent. This is done by defining mutual translations between relational and neighborhood models. Precisely speaking, not every neighborhood model has a relational representation. What we actually do is to define "normal" neighborhood models, and show that each normal neighborhood model can be transformed into an equivalent relational model. The converse direction is easier: any relational model can be transformed into an equivalent normal neighborhood model. As an immediate consequence of these translations, we can see that relational semantics and normal neighborhood semantics define the same logic.

For subject (2), we will consider a certain classical monotone modal logic, and show that it has a relational semantics (although it is not in the scope of the usual Kripke semantics), and it can be embedded into $\mathrm{S5} \otimes \mathrm{K}$, a classical normal bimodal logic with S5 and K modalities. First, we will observe that a classical neighborhood model for monotone modal logic can be regarded as a special case of our neighborhood model. This is done by regarding a classical neighborhood frame $\langle W, N\rangle$ as $\langle W, =, N\rangle$. In other words, a classical neighborhood frame is just an intuitionistic one whose $\leq$-part is degenerate. Under this identification, we apply a translation given in 1. to classical neighborhood models. This derives a relational representation of a certain classical monotone modal logic. Next, we use the fact that a relational model is also a model of $\mathrm{S5} \otimes \mathrm{K}$, which is easy to verify. This observation, together with the relational representation, induces a translation from our classical monotone modal logic to $\mathrm{S5} \otimes \mathrm{K}$.

5.1.3 Organization of the Chapter

Section 5.2 introduces neighborhood semantics, and give a sound and complete axiomatization. The resulting logic is slightly weaker than the logic introduced in Chapter 2. After seeing this, we will define normal neighborhood models, and show that they determine the same logic as the relational semantics.

In Section 5.3 we will give translations between relational and normal neighborhood models, and show that these translations do not change the interpretation of formulas in an appropriate sense.

In Section 5.4, we will consider a certain classical monotone modal logic. We first introduce a neighborhood semantics. The logic turns out to be a classical variant of the logic considered in Section 5.2. We will define relational semantics for the classical monotone modal logic, and establish an embedding into a classical bimodal logic $\mathrm{S5} \otimes \mathrm{K}$.

Finally, in Section 5.5 we summarize the chapter, and discuss relationship between our work and existing approaches.

5.2 Neighborhood Semantics

5.2.1 Definition of Neighborhood Semantics

Classically a neighborhood frame is given by a pair $\langle W, N\rangle$, where W is a set of possible worlds and N is a map from W to $\mathcal{P}(\mathcal{P}(W))$ [12, Part III], and is called a neighborhood function. Here we consider its intuitionistic version, so we introduce additional relation $\leq$.

Definition 5.2.1. An *intuitionistic neighborhood frame* (IN-frame, for short) is a triple $\langle W, \leq, N\rangle$ of a non-empty set W, a preorder $\leq$ on W, and a mapping $N : W \to \mathcal{P}(\mathcal{P}(W))$ that satisfies the *decreasing condition*:

$$x \leq y \implies N(x) \supseteq N(y).$$

The notion of a valuation is defined in the same way as the relational case.

Definition 5.2.2. For an IN-frame $\mathcal{N} = \langle W, \leq, N\rangle$, an $\mathcal{N}$*-valuation* is a map V from PV to $\mathcal{P}(W)$. An $\mathcal{N}$-valuation V is said to be *admissible* if $V(p)$ is upward-closed for all $p \in \mathrm{PV}$.

Definition 5.2.3. An *intuitionistic neighborhood model* (IN-model) is a pair $\langle \mathcal{N}, V\rangle$ of an IN-frame $\mathcal{N}$ and an admissible $\mathcal{N}$-valuation V.

Definition 5.2.4. Given an IN-model $\langle \mathcal{N}, V\rangle$, we can define the satisfaction relation, denoted by $\Vdash_{\mathrm{n}}$, in the same way as in the relational case, except that the truth conditions of modalities read

$$\begin{aligned} \mathcal{N}, V, x \Vdash_{\mathrm{n}} \Box A &\iff \forall X \in N(x).\forall y \in X.\mathcal{N}, V, y \Vdash_{\mathrm{n}} A; \\ \mathcal{N}, V, x \Vdash_{\mathrm{n}} \Diamond A &\iff \forall X \in N(x).\exists y \in X.\mathcal{N}, V, y \Vdash_{\mathrm{n}} A. \end{aligned} \tag{5.1}$$

The notion of truth in a model and a class of models is defined in the same way as Definition 2.4.7.

Remark 5.2.5. The conditions (5.1) are different from the ones in the usual neighborhood semantics, which read

$$\begin{aligned} \mathcal{N}, V, x \Vdash_{\mathrm{n}} \Box A &\iff \exists X \in N(x). \forall y.(y \in X \iff \mathcal{N}, V, y \Vdash_{\mathrm{n}} A); \\ \mathcal{N}, V, x \Vdash_{\mathrm{n}} \Diamond A &\iff \forall X \in N(x). \exists y. \neg(y \in X \iff \mathcal{N}, V, y \Vdash_{\mathrm{n}} A). \end{aligned} \tag{5.2}$$

This difference is motivated from our goal, that is, to establish a model of a modal logic with normal $\Box$ and non-normal $\Diamond$. In particular, unlike the usual classical modal logic, $\Box$ and $\Diamond$ cannot be each other's dual.

5.2.2 Canonical Model Construction

Next we are going to prove some completeness results for the neighborhood semantics introduced above, but before doing that we will prove a lemma for later use.

What we are going to do is to establish an analogue of the notion of canonicity known in the classical modal logic [8]. Basic outline of the current construction and proofs are the same as the previously known one, except that we consider neighborhood frames instead of Kripke frames.

Below, we will construct a canonical model for a logic Λ and prove standard properties. In what follows, we fix an arbitrary logic Λ containing $\mathrm{IK}^- + \mathbf{N}_{\Diamond\Box}$.

Definition 5.2.6 (canonical neighborhood frame)**.** We define $\mathcal{N}^\Lambda$ to be a tuple $\langle W^\Lambda, \leq^\Lambda, N^\Lambda \rangle$, where

- W^Λ is the set of all consistent prime Λ-theories,
- $\leq^\Lambda$ is the inclusion relation between sets, and
- N^Λ is defined by

$$\begin{aligned} N^\Lambda(\Gamma) &= \left\{ n(\Gamma', A) \,\middle|\, \Gamma \subseteq \Gamma' \in W^\Lambda, \Diamond A \notin \Gamma' \right\}, \\ n(\Gamma, A) &= \left\{ \Delta \in W^\Lambda \,\middle|\, \Box^{-1}\Gamma \subseteq \Delta \text{ and } A \notin \Delta \right\} \end{aligned}$$

  for each $\Gamma \in W^\Lambda$.

Definition 5.2.7 (canonical valuation)**.** A valuation V^Λ on $\mathcal{N}^\Lambda$ is defined by

$$V^\Lambda(p) = \{ \Gamma \mid p \in \Gamma \}.$$

Definition 5.2.8 (canonical model)**.** Clearly V^Λ is admissible, so $\mathcal{M}^\Lambda = \langle \mathcal{N}^\Lambda, V^\Lambda \rangle$ is an IN-model. We call this model a *canonical model for* Λ.

As usual, the following holds.

Theorem 5.2.9. $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} A$ *if and only if* $A \in \Gamma$.

Proof. We proceed by induction on A. The cases when A is an atomic formula, $\bot$, conjunction, and disjunction are trivial.

$A \to B \in \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} A \to B$: For all $\Delta \geq^\Lambda \Gamma$, if $\mathcal{N}^\Lambda, V^\Lambda, \Delta \Vdash_{\mathrm{n}} A$, then $A \in \Delta$ by induction hypothesis, so $A, A \to B \in \Delta$. Since Δ is closed under modus ponens, we obtain $B \in \Delta$. This means that $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} B$ by induction hypothesis.

$A \to B \notin \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} A \to B$: Suppose $A \to B \notin \Gamma$. Then, by deduction theorem and extension lemma, there exists $\Delta \in W^\Lambda$ such that $A \in \Delta$, $\Gamma \subseteq \Delta$, and $B \notin \Delta$. So $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} A \to B$.

$\Box A \in \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} \Box A$: If $\Delta \in n(\Gamma', B) \in N^\Lambda(\Gamma)$ for some Γ' and B, then $\Box^{-1}\Gamma \subseteq \Box^{-1}\Gamma' \subseteq \Delta$. Since $\Box A \in \Gamma$, we have $A \in \Box^{-1}\Gamma \subseteq \Delta$. Therefore $\mathcal{N}^\Lambda, V^\Lambda, \Delta \Vdash_{\mathrm{n}} A$. Since this holds for all $Gamma'$, B and Δ, it follows that $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} \Box A$.

$\Box A \notin \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} \Box A$: First, note that $\Box A \notin \Gamma$ means $\Diamond\bot \notin \Gamma$, since

$$\Diamond\bot \to \Box\bot, \Box\bot \to \Box A \in \Gamma.$$

From $\mathbf{N}_{\Diamond\Box}$ and the monotonicity of $\Box$. This means that $n(\Gamma, \bot) \in N^\Lambda(\Gamma)$. So it suffices to show that $n(\Gamma, \bot)$ contains some Δ such that $A \notin \Delta$. Such Δ can be obtained as follows. Since $\Box A \notin \Gamma$, we have $A \notin \Box^{-1}\Gamma$. By using extension lemma we can obtain a prime Λ-theory $\Delta \supseteq \Box^{-1}\Gamma$ with $A \notin \Delta$. For such Δ, it holds that $\Delta \in n(\Gamma, \bot)$.

$\underline{\Diamond A \in \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} \Diamond A}$: Take an arbitrary $n(\Gamma', B) \in N^\Lambda(\Gamma)$. Then we have $\Diamond B \notin \Gamma'$. Let Θ be the least theory containing $\Box^{-1}\Gamma'$ and A.

We first show that $B \notin \Theta$. If $B \in \Theta$, then we would have $A \to B \in \Box^{-1}\Gamma'$ from deduction theorem. This means $\Box(A \to B) \in \Gamma'$, hence $\Diamond A \to \Diamond B \in \Gamma'$ since

$$\Box(A \to B) \to \Diamond A \to \Diamond B \in \Gamma'.$$

However, $\Diamond A \in \Gamma \subseteq \Gamma'$ from assumption, so it follows that $\Diamond B \in \Gamma'$, a contradiction.

Now we have $\Box^{-1}\Gamma' \subseteq \Theta$, $A \in \Theta$, and $B \notin \Theta$. By extension lemma, there exists Δ satisfying the same conditions. For such Δ, we have $\Delta \in n(\Gamma', B)$, and $A \in \Delta$, and hence $\mathcal{N}^\Lambda, V^\Lambda, \Delta \Vdash_{\mathrm{n}} A$.

So we have proved that for all neighborhood $n(\Gamma', B)$ of Γ there exists $\Delta \in n(\Gamma', B)$ such that $\mathcal{N}^\Lambda, V^\Lambda, \Delta \Vdash_{\mathrm{n}} A$. This means that $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} \Diamond A$.

$\underline{\Diamond A \notin \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} \Diamond A}$: Assume $\Diamond A \notin \Gamma$, and let $X = n(\Gamma, A)$. Clearly X is a neighborhood of Γ, and any of its element Δ does not contain A. This means that $\mathcal{N}^\Lambda, V^\Lambda, \Delta \nVdash_{\mathrm{n}} A$ for all $\Delta \in X$. Therefore $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} \Diamond A$. □

The following is an easy consequence of this theorem.

Lemma 5.2.10. *Let $\mathcal{K}$ be a class of IN-models. If $\mathcal{M}^\Lambda \in \mathcal{K}$, then Λ is complete with respect to $\mathcal{K}$, that is, any formula true in $\mathcal{K}$ is a theorem of Λ.*

5.2.3 Sound and Complete Axiomatization

In the previous chapter, $\mathrm{IK}^- + \mathbf{N}_\Diamond$ is shown to be sound and complete for (non-fallible) relational semantics. Unfortunately, the same logic is not sound for the neighborhood semantics. This is because the axiom $\mathbf{N}_\Diamond$ ($\Diamond\bot \to \bot$) is not necessarily true in an intuitionistic neighborhood model (for a counterexample, see Lemma 5.2.13). However, there is an alternative of $\mathbf{N}_\Diamond$: by replacing $\mathbf{N}_\Diamond$ with $\mathbf{N}_{\Diamond\Box}$, we obtain a sound and complete axiomatization of neighborhood semantics.

Theorem 5.2.11 (Soundness and Completeness)**.** *$A \in \mathbf{L}(\Box, \Diamond)$ is a theorem of $\mathrm{IK}^- + \mathbf{N}_{\Diamond\Box}$ if and only if A is true in all IN-models.*

Proof. "Only if" part is proved in the usual way. For the converse direction, use Lemma 5.2.10 and the fact that the canonical model of $\mathrm{IK}^- + \mathbf{N}_{\Diamond\Box}$ is an IN-model, which has already been mentioned above. □

5.2.4 Normal Models

Although the logic determined by the neighborhood semantics does not coincide with $\mathrm{IK}^- + \mathbf{N}_\Diamond$ (the logic of relational semantics), there exists a class of IN-frames that characterizes $\mathrm{IK}^- + \mathbf{N}_\Diamond$.

First, we introduce the notion of normal worlds, at which the axiom $\mathbf{N}_\Diamond$ holds.

Definition 5.2.12. Let $\mathcal{N} = \langle W, \leq, N\rangle$ be an IN-frame.

1. A possible world $x \in W$ is said to be *normal* if $N(x) \neq \emptyset$.
2. $\mathcal{N}$ is said to be *normal* if every world $x \in W$ is normal.
3. An IN-model $\langle \mathcal{N}, V\rangle$ is said to be *normal* if $\mathcal{N}$ is normal.

The key observation is the following lemma.

Lemma 5.2.13. *Let $\langle \mathcal{N}, V\rangle$ be an IN-model. Then, a world $x \in W$ is normal if and only if it satisfies $\mathcal{N}, V, x \Vdash_{\mathrm{n}} \neg\Diamond\bot$ (not depending on the choice of V).*

Lemma 5.2.14. *The canonical model of $\mathrm{IK}^- + \mathbf{N}_\Diamond$ is normal.*

Proof. It suffices to show that $N^{\mathrm{IK}^- + \mathbf{N}_\Diamond}(\Gamma) \neq \emptyset$ for each Γ. Actually, we can show that $n(\Gamma, \bot)$ is always a neighborhood of Γ. This follows from the presence of $\mathbf{N}_\Diamond$: any prime Γ does not contain $\Diamond\bot$ since $\neg\Diamond\bot \in \Gamma$ and $\bot \notin \Gamma$. □

Theorem 5.2.15. *A formula A is a theorem of* $\mathrm{IK}^- + \mathbf{N}_\Diamond$ *if and only if A is true in all normal IN-models.*

Proof. "Only if" part is verified from the previous lemma and Theorem 5.2.11. For the converse, put Lemma 5.2.10 and Lemma 5.2.14 together. ☐

5.3 Translations Between Semantics

In the previous section, we have defined a neighborhood semantics, and observed that normal neighborhood frames correspond to the logic determined by relational semantics. In this section, we will give mutual translations between relational models and normal neighborhood models.

We first consider a translation from relational to normal neighborhood models, and then consider the converse direction. Both of the translations are proved to "preserve semantics" in an appropriate sense. These translations explicitly relate relational and neighborhood approach.

5.3.1 From Relational to Neighborhood Semantics

Definition 5.3.1. Let $\mathcal{R} = \langle W, \leq, R\rangle$ be an IR-frame. Then we define the IN-frame induced from $\mathcal{R}$ to be the tuple $\mathcal{N}_\mathcal{R} = \langle W, \leq, N_R\rangle$, where

$$N_R(x) = \{R\,[y] \mid y \geq x\}\,.$$

It is easy to check that $\mathcal{N}_\mathcal{R}$ is a normal IN-frame for any IR-frame $\mathcal{R}$. Additionally, an $\mathcal{R}$-valuation and an $\mathcal{N}_\mathcal{R}$-valuation are the same thing since they are both a mapping from PV to $\mathcal{P}(W)$. Admissibility of these valuations also coincide, since $\mathcal{R}$ and $\mathcal{N}_\mathcal{R}$ has the same preorder structure. To summarize, the following lemma holds:

Lemma 5.3.2. *Let $\langle \mathcal{R}, V\rangle$ be an IR-model. Then $\langle \mathcal{N}_\mathcal{R}, V\rangle$ is a normal IN-model.*

Theorem 5.3.3. *Let $\langle \mathcal{R}, V\rangle$ be an IR-model. Then,*

$$\mathcal{R}, V, x \Vdash_\mathrm{r} A \iff \mathcal{N}_\mathcal{R}, V, x \Vdash_\mathrm{n} A.$$

Proof. By induction on A. ☐

5.3.2 From Neighborhood to Relational Semantics

Definition 5.3.4. Let $\mathcal{N} = \langle W, \leq, N\rangle$ be an IN-frame. Then we define an IR-frame induced from $\mathcal{N}$ as a tuple $\mathcal{R}_\mathcal{N} = \langle \widetilde{W}, \widetilde{\leq}, \widetilde{\ni}\rangle$, where

$$\begin{aligned} \widetilde{W} &= \{(x, X) \mid x \in W, X \in N(x)\}\,; \\ (x, X)\ \widetilde{\leq}\ (y, Y) &\iff x \leq y; \\ (x, X)\ \widetilde{\ni}\ (y, Y) &\iff y \in X. \end{aligned} \tag{5.3}$$

It is clear that $\mathcal{R}_\mathcal{N}$ is an IR-frame for any IN-frame $\mathcal{N}$. This time, unlike the previous case, we need a little more consideration on valuations, since the set of possible worlds has been changed.

Definition 5.3.5. Let $\mathcal{N}$ be an IN-frame, and V an $\mathcal{N}$-valuation. Then we define an $\mathcal{R}_\mathcal{N}$-valuation $\widetilde{V}$ by

$$\widetilde{V}(p) = \left\{(x, X) \in \widetilde{W} \;\middle|\; x \in V(p)\right\}.$$

Lemma 5.3.6. *Let $\mathcal{N}$ be an IN-frame, and V an $\mathcal{N}$-valuation. If V is admissible, then $\widetilde{V}$ is admissible. Therefore, if $\langle \mathcal{N}, V\rangle$ is an IN-model, then $\langle \mathcal{R}_\mathcal{N}, \widetilde{V}\rangle$ is an IR-model.*

If $\mathcal{N}$ is normal, this transformation preserves semantics in the following sense:

Theorem 5.3.7. *Let $\langle \mathcal{N}, V\rangle$ be a normal IN-model. Then the following are equivalent:*

1. $\mathcal{N}, V, x \Vdash_{\mathrm{n}} A$*;*

2. $\mathcal{R}_{\mathcal{N}}, \widetilde{V}, (x, X) \Vdash_{\mathrm{r}} A$ *for all* $X \in N(x)$*;*

3. $\mathcal{R}_{\mathcal{N}}, \widetilde{V}, (x, X) \Vdash_{\mathrm{r}} A$ *for some* $X \in N(x)$*.*

Proof. By induction on A. □

5.4 Application to the Classical Case

In the previous section, we have investigated the relationship between relational and neighborhood semantics. In this section, we apply the translation given there to the classical setting. As a result, we can obtain a relational representation of neighborhood semantics for classical monotone modal logics. As a consequence, we can also obtain an embedding from classical version of $\mathrm{IK}^- + \mathbf{N}_\Diamond + \mathbf{PEM}$ to a certain classical bimodal logic.

5.4.1 Classical Monotone Modalities

First of all, we briefly introduce classical modal logic with both normal and non-normal modalities. The formulation here is basically taken from the course note by Pacuit [29].

We consider the language $\mathbf{L}([\,], [\,\rangle, \langle\,], \langle\,\rangle)$.

Definition 5.4.1. A *classical neighborhood frame* (CN-frame, for short) is a pair $\langle W, N\rangle$, where W is a non-empty set and N is a map from W to $\mathcal{P}(\mathcal{P}(W))$.

Definition 5.4.2. For a CN-frame $\mathcal{N} = \langle W, N\rangle$, an $\mathcal{N}$*-valuation* is a map from PV to $\mathcal{P}(W)$. A classical neighborhood model (CN-model) is a pair $\langle \mathcal{N}, V\rangle$ of a CN-frame $\mathcal{N}$ and an $\mathcal{N}$-valuation V.

In a similar way to the intuitionistic case, we can define the truth of formulas by the following clauses:

$$\begin{aligned}
\mathcal{N}, V, x \Vdash [\,]A &\iff \forall X \in N(x).\forall y \in X.\mathcal{N}, V, y \Vdash A;\\
\mathcal{N}, V, x \Vdash [\,\rangle A &\iff \forall X \in N(x).\exists y \in X.\mathcal{N}, V, y \Vdash A;\\
\mathcal{N}, V, x \Vdash \langle\,]A &\iff \exists X \in N(x).\forall y \in X.\mathcal{N}, V, y \Vdash A;\\
\mathcal{N}, V, x \Vdash \langle\,\rangle A &\iff \exists X \in N(x).\exists y \in X.\mathcal{N}, V, y \Vdash A.
\end{aligned}$$

Propositional connectives are interpreted in the same way as the usual classical logic.

Remark 5.4.3.
1. $\langle\,]$ and $[\,\rangle$ can be regarded as $\Box$ and $\Diamond$ in classical monotone modal logics. So the $\mathbf{L}(\langle\,], [\,\rangle)$-fragment of the logic above is essentially the classical monotone modal logic, often called M or EM [12, Section 8.2].

2. $[\,]$ and $\langle\,\rangle$ are $\Box$ and $\Diamond$ in normal modal logic, that is, they satisfy the following axioms:

$$[\,](p \to p), \quad [\,](p \to q) \to [\,]p \to [\,]q,$$
$$\neg\langle\,\rangle\bot, \quad \langle\,\rangle(p \vee q) \to \langle\,\rangle p \vee \langle\,\rangle q.$$

   Therefore, the $\mathbf{L}([\,], \langle\,\rangle)$-fragment of the logic above is the minimal normal modal logic K.

5.4.2 Relational Semantics for the Classical Setting

The classical modal logic defined above can be regarded as a special case of the neighborhood semantics defined in Section 5.2. This fact, combined with the result of Section 5.3, suggests that we can define a relational semantics for a monotone modal logic. Below we will see how to define a relational semantics for the classical monotone modal logic.

Remark 5.4.4. The semantics on CN-frame $\langle W, N\rangle$ introduced in the current section is the same as the semantics on $\langle W, =, N\rangle$ defined in Section 5.2.

To see this, we have to identify formulas of $\mathbf{L}([\,], [\,\rangle, \langle\,], \langle\,\rangle)$ and of $\mathbf{L}(\Box, \Diamond)$, and regard a CN-model as an IN-model.

First, we translate formulas from $\mathbf{L}([\,], [\,\rangle, \langle\,], \langle\,\rangle)$ to $\mathbf{L}(\Box, \Diamond)$ being based on the observation in Remark 5.4.3. We translate $[\,]$ and $[\,\rangle$ into $\Box$ and $\Diamond$, respectively. Other two modalities $\langle\,]$ and $\langle\,\rangle$ are duals of $[\,\rangle$ and $[\,]$, so they can be defined by $\neg\Diamond\neg$ and $\neg\Box\neg$, respectively.

As for models, we can regard $\langle W, N\rangle$ as $\langle W, =, N\rangle$, where $=$ is the equality on the set W. Since any valuation is admissible in $\langle W, =, N\rangle$, valuations on $\langle W, N\rangle$ and valuations on $\langle W, =, N\rangle$ are the same. It is easy to see that

$$\langle W, N\rangle, V, x \Vdash A \iff \langle W, =, N\rangle, V, x \Vdash_{\mathrm{n}} A',$$

where $A \in \mathbf{L}([\,], [\,\rangle, \langle\,], \langle\,\rangle)$, and $A' \in \mathbf{L}(\Box, \Diamond)$ is its correspondent as described above.

Based on this observation, in what follows, we consider $[\,]$ and $[\,\rangle$ as the only primitive modalities, and denote them by $\Box$ and $\Diamond$.

Let $\langle W, N\rangle$ be a CN-frame, and regard this as $\langle W, =, N\rangle$. Then we can apply the transformation (5.3) to obtain $\langle \widetilde{W}, \widetilde{\leq}, \widetilde{\ni}\rangle$. Here $\widetilde{\leq}$ is given by

$$(x, X) \,\widetilde{\leq}\, (y, Y) \iff x = y,$$

so $\widetilde{\leq}$ becomes an equivalence relation.

The definition of the interpretation of formulas is the same as in Chapter 2. Since $\widetilde{\leq}$ is an equivalence relation, and an interpretation is hereditary, the semantics is defined modulo this equivalence.

By abstracting these observations, we obtain the following definition.

Definition 5.4.5. 1. An IR-frame $\langle W, \simeq, R\rangle$ is said to be *degenerate* if $\simeq$ is an equivalence relation.

2. An IR-model $\langle \mathcal{R}, V\rangle$ is said to be *degenerate* if $\mathcal{R}$ is degenerate.

Since a degenerate IR-frame is just a special case of an IR-frame, we can interpret modal formulas in this frame in the same way as the intuitionistic case.

A sound and complete axiomatization is obtained by adding the principle of excluded middle to the intuitionistic case.

Theorem 5.4.6. *A formula $A \in \mathbf{L}(\Box, \Diamond)$ is a theorem in* $\mathrm{IK}^- + \mathbf{PEM} + \mathbf{N}_\Diamond$ *if and only if it is true in all degenerate IR-models.*

Below we will prove this theorem.

Here, we identify a CN-frame $\langle W, N\rangle$ and an IN-frame $\langle W, =, N\rangle$, and similarly for a CN-model and an IN-model. Then the following is immediate from Lemma 5.2.10.

Lemma 5.4.7. *A formula $A \in \mathbf{L}(\Box, \Diamond)$ is a theorem of* $\mathrm{IK}^- + \mathbf{PEM} + \mathbf{N}_\Diamond$ *if and only if it is true in all normal CN-models.*

Proof. It is straightforward to verify that the canonical model of $\mathrm{IK}^- + \mathbf{PEM} + \mathbf{N}_\Diamond$ is a normal CN-model. Therefore, completeness follows from Lemma 5.2.10. Soundness is proved in the usual way. $\Box$

So it suffices to prove that degenerate IR-models and normal CN-models determine the same logic. Its proof is based on the translations between models presented in Section 5.3, but there is a subtle problem. If $\langle \mathcal{N}, V\rangle$ is a classical neighborhood model, then its translation is a degenerate IR-model, so this direction is straightforward. Consider the other direction. If we have a degenerate IR-model $\langle \mathcal{R}, V\rangle$, by translation we obtain an IN-model $\langle \mathcal{N}_{\mathcal{R}}, V\rangle$, which is not necessarily classical. A neighborhood frame is classical when its $\leq$-part is the equality, but here this is not the case (it is only an equivalence relation).

Actually, this is not a big problem. We can fix it by considering quotient of $\mathcal{N}_{\mathcal{R}}$, which is indeed a normal CN-frame. In general, we can prove the following:

Proposition 5.4.8. *Let $\mathcal{N} = \langle W, \leq, N\rangle$ be an IN-frame, and V an admissible $\mathcal{N}$-valuation. Define their quotients $\hat{\mathcal{N}} = \langle \hat{W}, \hat{\leq}, \hat{N}\rangle$ and $\hat{V}$ as follows.*

- $\hat{W} = W/{\sim}$, *where $x \sim y$ if and only if $x \leq y$ and $y \leq x$.*
- $[x] \mathrel{\hat{\leq}} [y]$ *if and only if $x \leq y$, where $[z]$ denotes the equivalence class of z. This does not depend on the choice of x and y.*

- $\hat{N}([x]) = \{X/{\sim} \mid X \in N(x)\}$, *where* $X/{\sim}$ *is the image of* X *under the canonical projection* $W \to \hat{W}$. *Since* N *is decreasing,* $x \sim y$ *implies* $N(x) = N(y)$, *so* $\hat{N}$ *is well-defined.*

- $\hat{V}(p) = V(p)/{\sim}$.

Then, for any $A \in \mathbf{L}(\Box, \Diamond)$, *we have*

$$\mathcal{N}, V, x \Vdash_{\mathrm{n}} A \iff \hat{\mathcal{N}}, \hat{V}, [x] \Vdash_{\mathrm{n}} A.$$

Proof. By induction on A. □

For any IN-frame $\mathcal{N} = \langle W, \leq, N \rangle$, the preorder structure $\hat{\leq}$ of its quotient $\hat{\mathcal{N}}$ is clearly an order, that is, it is antisymmetric. In particular, when $\leq$ is an equivalence relation, $\hat{\leq}$ is the equality on $\hat{W}$. Normality of models is also preserved by quotient. Therefore applying this construction to $\mathcal{N}_{\mathcal{R}}$, we can see that each degenerate IR-model has an equivalent normal CN-model $\hat{\mathcal{N}}_{\mathcal{R}}$. This completes the proof of Theorem 5.4.6.

5.4.3 An Embedding into $\mathrm{S5} \otimes \mathrm{K}$

Relational semantics for monotone modal logic given above suggests that the monotone modal logic can be embedded into a normal bimodal logic. Since a degenerate IR-model has two binary relations corresponding to S5 and K, it is natural to think of a translation from the monotone modal logic to a bimodal logic $\mathrm{S5} \otimes \mathrm{K}$.

A frame for $\mathrm{S5} \otimes \mathrm{K}$ is just a degenerate IR-frame. However, a notion of valuation is not the same.

Definition 5.4.9. A $(\mathrm{S5} \otimes \mathrm{K})$-model is a pair $\langle \mathcal{R}, V \rangle$ of a degenerate IR-frame and a (not necessarily admissible) valuation V.

Given an $(\mathrm{S5} \otimes \mathrm{K})$-model, we can define a satisfaction relation, denoted by $\Vdash_{\mathrm{b}}$. The boolean connectives are interpreted in the usual way, and modalities are interpreted as follows:

$$\begin{aligned} \mathcal{R}, V, x \Vdash_{\mathrm{b}} \Box_1 A &\iff \forall y.(x \simeq y \implies \mathcal{R}, V, y \Vdash_{\mathrm{b}} A); \\ \mathcal{R}, V, x \Vdash_{\mathrm{b}} \Box_2 A &\iff \forall y.(x \mathrel{R} y \implies \mathcal{R}, V, y \Vdash_{\mathrm{b}} A). \end{aligned}$$

Proposition 5.4.10. *The logic* $\mathrm{S5} \otimes \mathrm{K}$ *is axiomatized by the following axioms and inference rules:*

- *modus ponens;*
- *necessitation for both* $\Box_1$ *and* $\Box_2$*;*
- *classical tautology instances;*
- *axioms K, T, and 5 for* $\Box_1$*;*
- *axiom K for* $\Box_2$*.*

Proof. By the canonical model construction [8, Section 4.2]. □

Now we can define a translation from $\mathbf{L}(\Box, \Diamond)$ to $\mathbf{L}(\Box_1, \Box_2)$.

Definition 5.4.11. For each $A \in \mathbf{L}(\Box, \Diamond)$, define $|A|$ as follows:

$$\begin{gathered} |p| = \Box_1 p; \quad |\bot| = \Box_1 \bot; \\ |A * B| = |A| * |B|; \quad (* \text{ is either } \wedge, \vee, \text{ or } \to) \\ |\Box A| = \Box_1 \Box_2 |A|; \quad |\Diamond A| = \Box_1 \Diamond_2 |A|. \end{gathered}$$

Definition 5.4.12. Let V be a valuation on a degenerate IR-frame. Then its *admissible variant*, denoted by V°, is defined by

$$V^\circ(p) = \{x \mid \forall y \simeq x.\, y \in V(p)\}.$$

Lemma 5.4.13. *For all formulas $A \in \mathbf{L}(\Box, \Diamond)$ and a valuation V,*

$$\mathcal{R}, V, x \Vdash_{\mathrm{b}} |A| \iff \mathcal{R}, V^{\circ}, x \Vdash_{\mathrm{b}} |A| \iff \mathcal{R}, V^{\circ}, x \Vdash_{\mathrm{r}} A.$$

Proof. By induction on A. $\Box$

Theorem 5.4.14. *Let $A \in \mathbf{L}(\Box, \Diamond)$. Then, A is true in all degenerate IR-models if and only if $|A|$ is true in all* (S5 $\otimes$ K)*-models.*

Proof. Let us write $\mathcal{R}, V \Vdash_{\mathrm{r}} A$ if $\mathcal{R}, V, x \Vdash_{\mathrm{r}} A$ for all x, and similarly for $\mathcal{R}, V \Vdash_{\mathrm{b}} A$. Then the condition

$$A \text{ is true in all degenerate IR-models}$$

is rephrased as

$$\mathcal{R}, V \Vdash_{\mathrm{r}} A \text{ for any } \mathcal{R} \text{ and admissible } V,$$

where $\mathcal{R}$ ranges over all degenerate IR-frames, and V ranges over all $\mathcal{R}$-valuations. This is equivalent to

$$\mathcal{R}, V^{\circ} \Vdash_{\mathrm{r}} A \text{ for any } \mathcal{R} \text{ and (not necessarily admissible) } V,$$

since V° is admissible for every V, and every admissible valuation V is an admissible variant of some valuation (because $V = V^{\circ}$ if V is admissible). By using the previous lemma, we can rewrite this into

$$\mathcal{R}, V \Vdash_{\mathrm{b}} |A| \text{ for any } \mathcal{R} \text{ and (not necessarily admissible) } V,$$

and this means

$$|A| \text{ is true in all } (\mathrm{S5} \otimes \mathrm{K})\text{-models.}$$ $\Box$

5.5 Summary and Remarks

5.5.1 Summary

We have investigated semantic aspects of IMLs without the distributivity law $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$. We have defined neighborhood semantics, and proved that the existing relational semantics can be represented in terms of this neighborhood semantics, as well as the converse holds under a slight restriction. This suggests that there is a close relationship between these two semantics.

By using this result, we have also considered the relationship between classical monotone modal logic and normal bimodal logic. We proved that the classical monotone modal logic with $\mathbf{N}_{\Diamond}$ has a relational representation of its neighborhood semantics, and is embeddable into S5 $\otimes$ K.

The results obtained from these investigations brings us a new insight in existing IMLs and non-normal modal logics. In particular, it turned out that (some of) the existing IMLs can naturally be captured as intuitionistic versions of non-normal modal logics, rather than normal ones.

5.5.2 Non-Normal Modalities and Multimodal Logics

A translation from non-normal modal logics to normal multimodal logics has already been studied before. Gasquet and Herzig showed that non-normal (not necessarily monotone) modal logic can be translated into normal modal logic with three modalities [18]. Kracht and Wolter proved that monotone modal logic can be "simulated" by normal bimodal logic (actually, normal monomodal logic can also simulate monotone modal logics) [22].

The basic idea behind their work is different from ours. Our translation from monotone to bimodal logic is based on the idea of considering the set

$$\widetilde{W} = \{(x, X) \mid x \in W, X \in N(x)\},$$

which consists of all pairs of possible worlds and their neighborhoods. On the other hand, both of the two previous approaches consider the set

$$W \cup \bigcup_{x \in W} N(x),$$

which consists of all possible worlds and all subsets of W that are neighborhoods of some worlds.[1]

This causes the difference in source and target logics of translations. In our translation, both of the source and target are stronger logics than the previous work. We assume the axiom $\mathbf{N}_\Diamond$ in the source logic, and considered $\mathrm{S5} \otimes \mathrm{K}$ as a target. In the previous work, they did not assume an extra axiom like $\mathbf{N}_\Diamond$, and the target is $\mathrm{K} \otimes \mathrm{K}$ (in the case of Kracht and Wolter) or $\mathrm{K} \otimes \mathrm{K} \otimes \mathrm{K}$ (in the case of Gasquet and Herzig).

5.5.3 Relationship with Gödel Translation

Our translation from monotone modal logic to $\mathrm{S5} \otimes \mathrm{K}$ can be considered as a variant of Gödel translation. Wolter and Zakharyaschev [43] investigated an embedding from IML into classical normal bimodal logic. They defined a Gödel-style translation, denoted by t, from an IML (with $\Box$ as the only primitive modality) into $\mathrm{S4} \otimes \mathrm{K}$. Our translation $|\cdot|$ can be seen as a variant of theirs.

At first sight, there is a difference between these two translations in the case of implication. Wolter and Zakharyaschev's t is defined as

$$t(A \to B) = \Box_1(t(A) \to t(B)),$$

which is the same as the usual Gödel translation, while our version $|\cdot|$ is given by

$$|A \to B| = |A| \to |B| \, .$$

However, when $\Box_1$ is an S5 modality, this makes no difference; we can prove that $|A| \to |B|$ and $\Box_1(|A| \to |B|)$ are equivalent in $\mathrm{S5} \otimes \mathrm{K}$.

[1] Actually, Kracht and Wolter used a more sophisticated technique, but the basic idea is as described here.

Chapter 6

Discussion

We have developed several semantics for IMLs and have investigated the relationships among them. However, the meanings of these results are, compared with standard Kripke semantics, quite unclear. For example, what each element of W in an IR-frame represents? In this chapter, we discuss such questions and try to give an intuitive understanding of our technical results.

6.1 Introduction

6.1.1 Motivation

In the traditional Kripke semantics, $\Diamond$ is understood as a variant of existential quantification: $\Diamond A$ is true at x if and only if *there exists* some y accessible from x such that A is true at y. From this definition the distributivity law naturally follows, as a consequence of the theorem $\exists x.(\varphi \vee \psi) \to (\exists x.\varphi) \vee (\exists x.\psi)$, (the distributivity of $\exists$ over $\vee$, which can be proved in intuitionistic first-order logic).

However, in our relational semantics, the situation is not as simple. In intuitionistic first-order logic, the existential quantifier is interpreted without mentioning $\leq$ in Kripke semantics, but the truth condition for $\Diamond$ involves $\leq$ as well as R. As a result, the traditional connection between $\Diamond$ and $\exists$ has been lost in our relational semantics.

The justification for such an interpretation is that it rejects the distributivity law, and this approach was successful to some extent, that is, we could construct a model of IML without the distributivity law (and we also obtained several completeness results for this semantics). However, the modified version does not seem to provide an intuitive understanding of non-distributive $\Diamond$-modality.

Here a question arises: Are there any natural way of understanding such a $\Diamond$-modality? The semantics we have given above is somewhat artificial, and does not give an immediate answer. Below we will discuss how we can understand the technical results presented in the previous chapters. In particular, we discuss a natural reading of modalities that allows us to reject the distributivity law.

6.1.2 Overview of the Chapter

In Section 6.2, we introduce the notion of an internal and external observer, and we interpret $\Diamond$ as the possibility from an observer's viewpoint. If we take the external observer's view, the distributivity can be justified in a natural way. However, from the internal observer's view, the same justification does not work, and the distributivity law is implausible.

In Section 6.3 we mention a possibility of defining a model that reflects this point of view more directly. We consider a counterpart frame to define such a model.

After that, in Section 6.4 we consider another, a little more abstract way for explaining non-distributive $\Diamond$ by considering "uncertainty about other worlds." We consider how uncertainty about other worlds appears in our semantics. Although our relational and neighborhood semantics are not a direct representation of an internal observer's viewpoint, we can find a similarity between these semantics and the internal observer interpretation.

In Section 6.5 we discuss what a possible world represents in each of the semantics. We mention related work in Section 6.6, and summarize the chapter in Section 6.7.

6.2 Introducing the Notion of Observers

6.2.1 Internal and External Observers

In modal logic, we often say "A is true at x" (or $x \Vdash A$), but what is the precise meaning of this statement? Here we consider its two possible readings, and discuss the difference between them.

One possible interpretation of $x \Vdash A$ is "an observer at state x can confirm that A is true." Here we assume there is an observer assigned to each world, and when we speak about truth at x, its precise meaning is the truth for the observer assigned to x. We will call this interpretation *internal observer interpretation*. This is different from the usual reading of modalities, and as we will see later this interpretation refutes the distributivity law.

More precisely, the meaning of $\Box$ and $\Diamond$ in the internal observer interpretation is given as follows. $x \Vdash \Box A$ if an observer at x (which we call o_x) knows that A is true at every y accessible from x. Similarly, $x \Vdash \Diamond A$ if o_x knows that there exists y accessible from x where A becomes true. At first sight, this interpretation looks the same as the usual one. The point is that, because o_x is fixed to a certain world, o_x cannot have complete information about other worlds. Therefore o_x have several possibilities in mind about the actual situations of y, and if we say o_x knows A is true at y, it means that A is true in every situation o_x considers possible.

Another interpretation of $x \Vdash A$ is to assume a single observer outside the model, and truth is considered as the truth for this observer. Here the observer does not depend on the choice of x, and the observer can see the state of all worlds at the same time. This interpretation is more or less the same as the usual Kripke semantics (although usually such an observer is not mentioned explicitly). We will call this version *external observer interpretation*.

6.2.2 Validity of the Distributivity Law

Now we will show that the external observer interpretation implies the distributivity law, but the internal one does not. The argument below considers the case of modal possibility, but the distributivity for next-operator in LTL can be treated in the same manner. The only difference is that in the modal case we say "for some accessible world" whereas in the LTL case we should say "for the next time (which is uniquely determined)."

From the external observer's viewpoint, we can freely go back and forth between states, and inspect any other accessible worlds' state. Informations obtained in this way can be used to decide whether a proposition holds at a certain world. If we take this point of view we can justify the distributivity law as follows:

1. Let us assume $\Diamond(A \vee B)$ holds at the current world x.
2. Then we know that $A \vee B$ holds at some accessible world y.
3. So move to y and check which of A and B is actually the case.
4. Do the case analysis:
   (a) If A is true at y, we have $\Diamond A$ at x.
   (b) If B is true at y, we have $\Diamond B$ at x.
5. In either case we have $\Diamond A \vee \Diamond B$ at x.

This argument is actually nothing but a straightforward proof that the distributivity law holds in an ordinary Kripke model. This is natural because external observer interpretation is just another way of seeing the traditional Kripke semantics. Also note that this argument is constructive, and therefore valid in intuitionistic logic.

From the internal observer's viewpoint, however, this justification does not work because the third step fails. The strategy requires to move to another state, do the case analysis, and then come back to the original world, but an internal observer is assigned to a fixed world, and cannot move to other worlds. Therefore, in the internal setting, knowing that "either A or B holds at some y," does not necessarily mean that it is possible to say either "A at y" or "B at y."

In view of this distinction, the LTL and IK$^-$ (and its extensions) we have considered in this thesis can be seen as IMLs based on the internal observer's viewpoint. Therefore, to define their Kripke semantics appropriately, we had to encode the internal observer's knowledge in terms of possible worlds and accessibility relations. This gives an explanation of why the straightforward approaches (such as the functional frames for LTL, and the interpretation of $\Diamond$ as existential quantifier) do not work for the non-distributive cases.

6.2.3 Another Variant of Internal Observers

Another possible version of the internal observer interpretation is to consider an observer inside the frame, but not assigned to a fixed world. Instead, she can move between worlds along the accessibility relation R, that is, she can move from x to y if and only if $x\ R\ y$ holds. In particular, once she has moved from x to y (where $x\ R\ y$ holds), it is not allowed to go back to x unless $y\ R\ x$ is also the case. We will call an observer of this type an *unfixed* observer, and an internal observer considered above a *fixed* one.

With only a slight adjustment to the case of the fixed observer, we can see that the distributivity law does not hold in view of an unfixed observer. If an observer is in the state x and knows that $\Diamond(A \vee B)$ is true, then she would have some y accessible from x in which either A or B is true. However, to decide which is actually the case, the observer has to move to y, and after that she cannot go back to x. So the observer can say "$A \vee B$ at y," but not "$\Diamond A \vee \Diamond B$ *at* x."

6.3 Formulating Internal Observers

Here we will introduce a model based on the internal observer interpretation. We use so-called counterpart frames, which have originally been considered as a semantics for first-order modal logic [23, 11].

6.3.1 Models of Internal Observers

A counterpart frame is a tuple $\langle W, R, D, C\rangle$ where

- W is a non-empty set,
- R is a binary relation on W,
- D_x is a non-empty set for each $x \in W$, and
- $C_{x,y}$ is a relation between D_x and D_y for each $(x, y) \in R$.

If $a \in D_x$ is related to $b \in D_y$ by $C_{x,y}$, then b is said to be a y-counterpart of a.

As a semantics of first-order modal logic, this structure can be understood as follows. W and R is the same as the usual Kripke semantics: W is the set of possible worlds, and R is an accessibility relation on W. For each $x \in W$, the set D_x is the collection of all individual objects that are available in x. The set D_x may vary from world to world, and there are no constraints on how they are related. $C_{x,y}$ is called a counterpart relation, and $a\ C_{x,y}\ b$ (where $a \in D_x$ and $b \in D_y$) means that an object a of a world x has a "counterpart" b in another world y. An object $a \in D_x$ is not, in general, necessarily available in y, but the counterpart relation $C_{x,y}$ indicates some (possibly multiple) objects $b \in D_y$ as possible approximations of a in y.

We will consider each $x \in W$ as a world to which an internal observer can be assigned, and D_x as the set of observer's possible states of knowledge (as in the Kripke semantics for intuitionistic logic). Since we consider intuitionistic modal logic, each D_x is equipped with a preorder $\leq$. The counterpart relation $a\ C_{x,y}\ b$ represents the relation such that the state of knowledge b is consistent with a in the sense that the knowledge about y known in a is consistent with the actual state of b (such a constraint can arise from modalities; for example, if $\Box A$ is known to be true in a, then A has to be true in every y-counterpart b of a).

To apply this structure to the semantics of IML, we have to assume that W and R are common knowledge, that is, an observer knows the structure of the frame $\langle W, R\rangle$ she is involved in. It is also assumed that each observer knows her location. These assumptions may be controversial from several motivations, and derives some extra axioms (see the next subsection), but for simplicity we confine ourselves to this particular setting.

We consider the following interpretation of modalities:

$$x, a \Vdash_{\mathrm{c}} \Box A \iff \forall y \in R[x] . \forall b \in C_{x,y}[a] . y, b \Vdash_{\mathrm{c}} A; \tag{6.1}$$

$$x, a \Vdash_{\mathrm{c}} \Diamond A \iff \exists y \in R[x] . \forall b \in C_{x,y}[a] . y, b \Vdash_{\mathrm{c}} A. \tag{6.2}$$

The interpretation of $\Box$ is natural: $\Box A$ is true for $\mathtt{o}_x$ having the knowledge a if and only if A is true at all counterparts b of a.

Compared to (6.1), the meaning of the interpretation (6.2) would be less clear. Here we define $\Diamond A$ to be true in view of $\mathtt{o}_x$ having the knowledge a if and only if there exists y such that A is "necessarily" true at y, that is, true at every y-counterparts of a.

In this interpretation, $\mathtt{o}_x$ can choose a world y to move to, but not the actual state $b \in C_{x,y}[a]$. Because the set $C_{x,y}[a]$ represents the collection of all actual states in y considered possible by $\mathtt{o}_x$, this condition means that $\mathtt{o}_x$ has to check that A is true in all possible situations. This is natural because, to assert that A is true, we have to know that A is true in all possible actual situations.

For heredity, we need the following additional condition (assuming that D_x's are not necessarily pairwise disjoint):

$$a, a' \in D_x, a \leq a', x \mathrel{R} y \implies \exists y' \in R[x] . \forall b' \in C_{x,y'}[a'] . \exists b \in C_{x,y}[a] . b \leq b'.$$

The following simpler version is a sufficient condition for this:

$$a, a' \in D_x, a \leq a', x \mathrel{R} y \implies C_{x,y}[a'] \subseteq C_{x,y}[a].$$

This condition says that each $C_{x,y}$ is decreasing as a map from D_x to $\mathcal{P}(D_y)$ (ordered by inclusion).

6.3.2 Axiomatization

Axiomatization of this semantics is an open problem. At this moment we can only say that IK^- and several extra axioms are sound for this semantics. It is easy to check the soundness of IK^-. One of the extra sound axioms is $\Diamond\top \vee \neg\Diamond\top$, whose validity is easily seen from the following facts.

Lemma 6.3.1. *1. $x, a \Vdash_{\mathrm{c}} \Diamond\top$ if and only if $R[x] \neq \emptyset$;*

2. $x, a \Vdash_{\mathrm{c}} \neg\Diamond\top$ if and only if $R[x] = \emptyset$.

In short, the truth of $\Diamond\top$ is determined by the shape of $\langle W, R\rangle$, and independent of the structure of D_x. This result is intuitively understood as a consequence of the assumption that each observer knows her location and the structure of $\langle W, R\rangle$.

More detailed investigation on axiomatization, as well as other variants of the semantics (not assuming that the observers have perfect knowledge about $\langle W, R\rangle$, for example), is left for future work.

6.4 Uncertainty and Distributivity

We will now discuss the relationship between the notion of an internal observer and the semantics considered in the previous chapters. We argue that both of them express a kind of "uncertainty." Although direct connection between the internal observer interpretation and our relational/neighborhood semantics does not immediately follow from this observation, this relates mathematical model and rather informal interpretation to some extent.

6.4.1 Internal Observer's Uncertainty

An internal observer cannot freely access other worlds. This means that an internal observer's knowledge is not complete, that is, it does not contain all informations about other worlds. Consider two distinct worlds x and y, and an observer in x. Then, the observer $\mathtt{o}_x$, being unable to move to y to see if A is true at y, may fail to decide whether A is true at y or not, because $\mathtt{o}_x$ does not necessarily have sufficient information about y.

Therefore, each observer may have several possibilities in mind about the actual situation of other worlds. As an example, consider the situation in which an observer can verify $\Diamond(A \vee B)$ but neither $\Diamond A$ nor $\Diamond B$ (this is the situation discussed in Section 6.2 to refute the distributivity). Then the observer would know that there exists y where $A \vee B$ is true, but does not know which of the disjuncts is the case. So the observer has at least two possibilities in mind about the actual situation of y. In one situation A becomes true, and in the other B is true. The observer cannot decide which of these two possibilities is the actual one. This is what we mean by saying that there is an uncertainty in the knowledge of the observer.

6.4.2 Counterparts as Uncertainty

Counterpart frames can be seen as a fairly direct representation of the uncertainty in the above sense. Roughly speaking, each counterpart represents one of the possible situations the observer has in mind.

Each $x \in W$ represents a world, and an element of D_x represents a possible state of knowledge of the observer $\mathtt{o}_x$. Possible states of knowledge $a \in D_x$ and $b \in D_y$ are related by the counterpart relation $C_{x,y}$. We understand $a\ C_{x,y}\ b$ as "$\mathtt{o}_x$ with knowledge a considers b as one of the possible situations of the world y." So the set $C_{x,y}[a]$ represents the amount of uncertainty, and the condition that $C_{x,y}$ is decreasing is understood as follows: the amount of uncertainty decreases if the amount of knowledge increases.

Based on this understanding, we can formalize the argument about distributivity above in counterpart semantics. Consider worlds $x, y \in W$, a state $a \in D_x$ and its y-counterparts $b, b' \in D_y$. Moreover, suppose that $y, b \Vdash_{\mathrm{c}} A$ and $y, b' \Vdash_{\mathrm{c}} B$. Then these b and b' are exactly what we called "two possibilities about the actual situation of y" above. Indeed, this construction gives a model that refutes the distributivity law.

6.4.3 Neighborhoods as Uncertainty

In neighborhood semantics, each neighborhood represents one possible actual situation of other worlds, and therefore the uncertainty corresponds to the fact that a world may have more than one neighborhood.

In the case of the refutation of the distributivity, two possibilities (either A or B) are represented as two neighborhoods. Let us call such neighborhoods X_1 and X_2. Then there exists a world $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_1 \Vdash A$ and $x_2 \Vdash B$. Since the disjunct being true depends on the choice of a neighborhood, neither $\Diamond A$ nor $\Diamond B$ holds, while $\Diamond(A \vee B)$ is true.

The decreasing condition on neighborhood function introduced in Chapter 5 is understood in the same way as the case of counterpart frames, that is, if an observer gains more knowledge, then the amount of uncertainty decreases.

6.4.4 Uncertainty in Relational Semantics

Next, we will discuss how the uncertainty appears in the relational semantics. Here we make use of the translations between neighborhood and relational semantics given in Chapter 5.

According to the translation, we can regard a possible world in relational semantics as a pair (x, X) of a possible world x (in neighborhood semantics) and a neighborhood X of x. Combining this correspondence with the interpretation of neighborhood semantics discussed above, we can understand relational semantics in terms of the internal observer interpretation. A possible world in relational semantics actually consists of two pieces of information: the observer's state of knowledge (x) and one of the possibilities about the situations of other worlds the observer has in mind (X).

As an example, consider again the situation refuting the distributivity. Let x be a state of knowledge of some observer, and has two possible situations in mind. These two possibilities can be represented as neighborhoods X_1 and X_2 where some $x_1 \in X_1$ and $x_2 \in X_2$ satisfy $x_1 \Vdash A$ and $x_2 \Vdash B$, respectively. In relational semantics, two distinct worlds are involved in this situation: (x, X_1) and (x, X_2). The fact that the observer has to consider both of them is reflected to the truth condition

$$x \Vdash_{\mathrm{r}} \Diamond A \iff \forall y \geq x. \exists z. (y\ R\ z \text{ and } z \Vdash_{\mathrm{r}} A)$$

and the definition of $\leq$ that makes (x, X_1) and (x, X_2) equivalent (that is, both $(x, X_1) \leq (x, X_2)$ and $(x, X_1) \geq (x, X_2)$ hold).

6.5 What Does a Possible World Represent?

Above we have discussed the meaning of modalities. There is another, closely related question: what is the meaning of possible worlds? In the above developed semantics, it seems that we have encountered at least two different types of possible worlds. Counterpart frames represents this aspect directly. In a counterpart frame $\langle W, R, D, C\rangle$, elements W and D_x correspond to each of the two types. Elements of W are not directly observable, and the actual concrete information (truth of propositional variables) is carried by the elements of D_x.

Similar distinction can be found in the literature (briefly mentioned in [6, p. 6]), and following them we distinguish these two types by "worlds" and "situations." Worlds and situations correspond to elements of W and D_x, respectively, of a counterpart frame. A situation determines the concrete state of affairs, and in each world there can be multiple possible situations. So an actual state of affairs is not specified by a world only.

To avoid confusion, we also use the terminology "point" for what we previously called a possible world (an element of a set W of "worlds"). Applying these terminologies, we can say that a point in a usual Kripke model (of the form $\langle W, R, V\rangle$) represents a situation, not a world.

In our neighborhood semantics, as well as the usual Kripke semantics, each point is a situation because a valuation specifies exactly which proposition is true at each point, and consequently each point in a model represents a possible complete collection of informations characterizing some situation. So situations are present in our neighborhood semantics, but worlds are not made explicit. However, as a world is something like a collection of situations, it seems that a neighborhood can play a role similar to a world (although further investigation is needed to justify such a correspondence between worlds/situations and neighborhoods/points).

In relational semantics, a point can be regarded as a situation for the same reason, but it carries additional information. As we have discussed in the previous section, a point in relational semantics actually carries more information than that of neighborhood semantics: It also contains the information of points accessible from it. If we consider a neighborhood as a world, this fact suggests that a point in a relational model is a pair of a situation and a world accessible from that situation.

6.6 Related Work

6.6.1 Aucher's Internal Epistemic Logic

An idea similar to our internal observer can be found in Aucher's work on epistemic logic [5]. He investigated an epistemic logic based on the view of an agent inside the situation, rather than the usual view from outside the situation.

He considered three approaches to epistemic logics: internal, perfect external, and imperfect external. By "internal" he means that the truth is considered from the viewpoint of one of the agents involved in the situation. He calls the viewpoint from which the truth is considered as the *modeler*. So his internal approach considers the situation where the modeler is involved in the epistemic situation. In the external approaches, the situation being considered does not involve the modeler. The external approaches are split into two, namely perfect and imperfect ones. In the perfect external approach he assumes the modeler has perfect knowledge of the situation including agents' belief, while in the imperfect approach there may be uncertainty about agents' belief.

Aucher and we use the same word "internal," but actually our internal observer and a modeler in Aucher's internal approach are not exactly the same. In his internal approach, the modeler is a particular agent called Y, and his internal epistemic logic has a modal operator B_Y, denoting "Y believes that." So the modeler appears in the language of internal epistemic logic, whereas in our approach this is not the case.

Rather, imperfect external epistemic logic is more close to our internal observer's viewpoint. In this approach a modeler is not made explicit, but have uncertainty about the situation.

Technically there is also a similarity between Aucher's imperfect external model and the structure we have encountered in Section 5.4. Imperfect external model is defined as a disjoint sum of several pointed Kripke models (M, w_0). Each component (M, w_0) of an imperfect external model represents a possible actual situation (a state of beliefs of the agents) the modeler has in mind. This reflects the intuition that in the imperfect external approach the modeler is uncertain about the situation. On the other hand, in our relational representation of classical neighborhood semantics, we have considered a Kripke model equipped with an equivalence relation. This equivalence can be understood as a representation of uncertainty in observer's knowledge. The equivalence relation appearing here and the "disjoint sum" in Aucher's approach correspond to each other (more precisely, an equivalence class of relational semantics and a component of imperfect external model corresponds to each other), and both represent the "uncertainty."

6.6.2 Simpson's Approach and External Observer

The approach considered by Simpson [37] is closely related to our external observer interpretation. His main idea is to use Kripke semantics for intuitionistic first-order logic to define a semantics of IMLs. The point is that the modalities are interpreted in the same way as the quantifiers in first-order logic. In the classical setting this results in the standard Kripke semantics, and Simpson's approach is to do the same thing in intuitionistic first-order logic.

Simpson emphasized this point. In the abstract of his thesis [37] he writes:

> The standard theory arises from interpreting the semantic definitions in the ordinary meta-theory of informal classical mathematics. If, however, the same semantic definitions are interpreted in an intuitionistic meta-theory then the induced modal logics no longer satisfy certain intuitionistically invalid principles. This thesis investigates the *intuitionistic modal logics* that arise in this way.

Also, on page 60 of the thesis he defines a translation from modal formulas to first-order formulas (which is the same as so-called standard translation, and studied well in the classical setting [8]), and places it as "a precise definition of the desired intuitionistic modal logic."

In short, he considered the usual (normal) modal logics in an intuitionistic meta-theory. This corresponds to our external observer interpretation, because (as already mentioned) it is another way of seeing the standard Kripke semantics.

6.7 Summary

We have considered the notion of internal and external observer. They give rise to two different interpretations of modalities, and it turned out that the external observer interpretation admits the distributivity law, while the internal one does not. This gives a clear explanation of the difference between IMLs with and without the distributivity law.

Although our relational and neighborhood semantics do not mention any observers explicitly, they have a similar structure to the internal observer interpretation. An uncertainty about other worlds plays a central role in these semantics, and it appeared in relational, neighborhood, and counterpart semantics in different forms. We can understand the failure of the distributivity law as a consequence of this uncertainty.

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