Quotation, Evaluation and Metavariables

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$\lambda\mathbf{Q}$ and $\lambda\mathbf{q}$

We introduce two calculi of quotation.

- $\lambda\mathbf{Q}$ is an explicit calculus of quotation.
- $\lambda\mathbf{q}$ is an implicit calculus of quotation.

$\lambda\mathbf{Q}$

Hypothesis

A *hypothesis* is an expression of the form $x : A$ where x is a variable and A is a type. A hypothesis $x : A$ is also written as x^A. A hypothesis is also called a *declaration.*

A *hypothesis sequence* is a sequece of hypotheses:

$$x_1 : A_1, \ldots, x_m : A_m$$

where $x_1, \ldots, x_m$ are distinct variables. We use γ, δ etc. to denote hypothesis sequence. We write $\gamma - x^A$ for the result of removing the declaration x^A from γ.

A *hypothesis sequence sequence* (HSS) is a sequece of hypothesis sequence

$$\gamma_1; \ldots; \gamma_n$$

where each γ_i is a hypothesis sequence We use Γ, Δ etc. to denote hypothesis sequence sequence.

Types of $\lambda\mathbf{Q}$

$$A, B ::= K \mid A \Rightarrow B \mid (\gamma)\,[A]$$

K is a set of atomic types.

Typing Rules

Variable Reference

$$\frac{}{\gamma_1; \ldots; \gamma_n \vdash \hat{\ }^m x : A}\ \mathsf{varref}$$

where x^A is declared in γ_{n-m} and $\hat{\ }^m$ is a sequence of m-copies of $\hat{\ }$.

We give two examples below.

$$\frac{}{x : A, y : B; z : C; x : D \vdash \hat{\ }\hat{\ }x : A}\ \mathsf{varref}$$

$$\frac{}{x : A, y : B; z : C; x : D \vdash x : D}\ \mathsf{varref}$$

Rules for $\Rightarrow$

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x^A\,[b] : A \Rightarrow B} \Rightarrow I \qquad \frac{\Gamma \vdash b : A \Rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash b(a) : B} \Rightarrow E$$

Rules for (γ)

$$\frac{\Gamma; \gamma \vdash b : B}{\Gamma \vdash (\gamma)\,[b] : (\gamma)\,[B]}\ (\gamma) I$$

$$\frac{\Gamma \vdash b : (\gamma)\,[B] \quad \Gamma \vdash a : A}{\Gamma \vdash b(x = a) : (\gamma - x^A)\,[B]}\ \gamma E \qquad \frac{\Gamma \vdash b : ()\,[B]}{\Gamma \vdash [\![b]\!] : B}\ () E$$

In γE, the hypothesis x^A must be declared in γ.

- If $\Gamma \vdash a : A$ is derivable, then a is said to be a $\lambda\mathbf{q}$-*term under* Γ.
- A term is *closed* if it is a term under the empty HSS.
- $(\gamma)\,[b]$ is called the *quotation* of b with respect to γ.
- $b(x = a)$ is called a *meta application*.
- $[\![b]\!]$ is called the *evaluation* of b.

- *Free* variable reference is defined as usual.
- For each a such that
$$\Gamma \vdash a : A,$$
we can define $\hat{a}$ such that
$$\Gamma ; \gamma \vdash \hat{a} : A$$
holds for any γ. For example,
$$\hat{}(x\,((x : A)\,[\hat{x}\,(x)]\,)) \equiv \hat{x}\,((x : A)\,[\hat{\hat{x}}\,(x)]\,).$$
- $[x := a](b)$, the result of *substituting* x with a in b, is defined almost as usual. For example,
$$\begin{aligned} & [x := y + x](x + \llbracket (x : \mathsf{N})\,[x + \hat{x}]\,\rrbracket) \\ \equiv\ & (y + x) + \llbracket (x : \mathsf{N})\,[x + \hat{}(y + x)]\,\rrbracket \\ \equiv\ & (y + x) + \llbracket (x : \mathsf{N})\,[x + (\hat{y} + \hat{x})]\,\rrbracket . \end{aligned}$$

Reduction Rules

$$\frac{}{\lambda x^A\,\texttt{[}b\texttt{]}\,\texttt{(}a\texttt{)} \rightarrow [x := a](b)}\ \beta$$

$$\frac{}{\texttt{(}\gamma\texttt{)}\,\texttt{[}b\texttt{]}\,\texttt{(}x\ \texttt{=}\ a\texttt{)} \rightarrow \texttt{(}\gamma - x^A\texttt{)}\,\texttt{[}[x := \texttt{\^{}}a](b)\texttt{]}}\ Q\beta \qquad \frac{}{[\![\texttt{()}\,\texttt{[\^{}}b\texttt{]}]\!] \rightarrow b}\ \mathsf{eval}$$

$$\frac{b \rightarrow c}{\lambda x^A\,\texttt{[}b\texttt{]} \rightarrow \lambda x^A\,\texttt{[}c\texttt{]}} \qquad \frac{b \rightarrow c}{b\texttt{(}a\texttt{)} \rightarrow c\texttt{(}a\texttt{)}} \qquad \frac{a \rightarrow c}{b\texttt{(}a\texttt{)} \rightarrow b\texttt{(}c\texttt{)}}$$

$$\frac{b \rightarrow c}{\texttt{(}\gamma\texttt{)}\,\texttt{[}b\texttt{]} \rightarrow \texttt{(}\gamma\texttt{)}\,\texttt{[}c\texttt{]}}$$

$$\frac{b \rightarrow c}{b\texttt{(}x\ \texttt{=}\ a\texttt{)} \rightarrow c\texttt{(}x\ \texttt{=}\ a\texttt{)}} \qquad \frac{a \rightarrow c}{b\texttt{(}x\ \texttt{=}\ a\texttt{)} \rightarrow b\texttt{(}x\ \texttt{=}\ c\texttt{)}}$$

$$\frac{b \rightarrow c}{[\![b]\!] \rightarrow [\![c]\!]}$$

$$
\begin{aligned}
& [\![(x, y)\,[x + y]\,(x = 2)\,(y = 3)]\!] \\
\rightarrow\ & [\![(y)\,[2 + y]\,(y = 3)]\!] \\
\rightarrow\ & [\![()\,[2 + 3]]\!] \\
\rightarrow\ & 2 + 3
\end{aligned}
$$

$$
\begin{aligned}
& \lambda x\,[\,[\![(y)\,[\hat{}x + y]\,(y = 3)]\!]\,]\,(2) \\
\rightarrow\ & \lambda x\,[\,[\![()\,[\hat{}x + 3]]\!]\,]\,(2) \\
\rightarrow\ & \lambda x\,[x + 3]\,(2) \\
\rightarrow\ & 2 + 3
\end{aligned}
$$

Properties of $\lambda\mathbf{Q}$

- Subject Reduction
- Church-Rosser
- Strong Normalizablity

$\lambda\mathbf{q}$

Types of $\lambda\mathbf{q}$

$$A, B ::= K \mid A \Rightarrow B \mid `A'$$

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(Types of $\lambda\mathbf{Q}$: $\quad A, B ::= K \mid A \Rightarrow B \mid (\gamma)\,[A]$)

$$\frac{}{\gamma_1; \ldots; \gamma_n \vdash \char`^{m} x : A} \ \mathsf{varref}$$

where x^A is declared in γ_{n-m}.

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x^A \, [b] : A \Rightarrow B} \Rightarrow I \qquad \frac{\Gamma \vdash b : A \Rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash b(a) : B} \Rightarrow E$$

$$\frac{\Gamma; \gamma \vdash b : B}{\Gamma \vdash \text{‘}b\text{’} : \text{‘}B\text{’}} \ \Box I$$

$$\frac{\Gamma \vdash b : \text{‘}B\text{’} \quad \Gamma \vdash a : A}{\Gamma \vdash b(x = a) : \text{‘}B\text{’}} \ \Box E_1 \qquad \frac{\Gamma \vdash b : \text{‘}B\text{’}}{\Gamma \vdash [\![b]\!] : B} \ \Box E_2$$

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holds for any γ. For example,
$$\hat{}(x(\text{`}\hat{}x(x)\text{'})) \equiv \hat{}x(\text{`}\hat{}\hat{}x(x)\text{'}).$$
- $[x := a](b)$, the result of *substituting* x with a in b, is defined almost as usual. For example,
$$\begin{aligned} & [x := y + x](x + [\![\text{`}x + \hat{}x\text{'}]\!]) \\ \equiv\ & (y + x) + [\![\text{`}x + \hat{}(y + x)\text{'}]\!] \\ \equiv\ & (y + x) + [\![\text{`}x + (\hat{}y + \hat{}x)\text{'}]\!]. \end{aligned}$$

$$\frac{}{\lambda x^A\,[b]\,(a) \to [x := a](b)}\ \beta$$

$$\frac{\Gamma; \gamma \vdash b \quad \Gamma \vdash a : A \quad x^A \in \gamma}{\text{‘}b\text{’}\,(x = a) \to \text{‘}[x := \hat{}a](b)\text{’}}\ q\beta \qquad \frac{}{[\![\text{‘}\hat{}b\text{’}]\!] \to b}\ \mathsf{eval}$$

$$\frac{b \to c}{\lambda x^A\,[b] \to \lambda x^A\,[c]} \qquad \frac{b \to c}{b(a) \to c(a)} \qquad \frac{a \to c}{b(a) \to b(c)}$$

$$\frac{b \to c}{\lambda x^A\,[b] \to \lambda x^A\,[c]}$$

$$\frac{b \to c}{b(x = a) \to c(x = a)} \qquad \frac{a \to c}{b(x = a) \to b(x = c)}$$

$$\frac{b \to c}{[\![b]\!] \to [\![c]\!]}$$

$$
\begin{array}{rl}
 & [\![\, \text{‘}\, x + y \,\text{’}\, (x = 2)\, (y = 3) \,]\!] \\
\rightarrow & [\![\, \text{‘}\, 2 + y \,\text{’}\, (y = 3) \,]\!] \\
\rightarrow & [\![\, \text{‘}\, 2 + 3 \,\text{’} \,]\!] \\
\rightarrow & 2 + 3
\end{array}
$$

$$
\begin{array}{rl}
 & \lambda X\, [\lambda x\, [\lambda y\, [x + [\![\, X\, (x = x)\, (y = y) \,]\!]\,]\, (3)\,]\,]\, (\text{‘}\, x + y \,\text{’}) \\
\rightarrow & \lambda X\, [\lambda x\, [x + [\![\, X\, (x = x)\, (y = 3) \,]\!]\,]\,]\, (\text{‘}\, x + y \,\text{’}) \\
\rightarrow & \lambda x\, [x + [\![\, \text{‘}\, x + y \,\text{’}\, (x = x)\, (y = 3) \,]\!]\,] \\
\rightarrow & \lambda x\, [x + [\![\, \text{‘}\, \hat{}\, x + y \,\text{’}\, (y = 3) \,]\!]\,] \\
\rightarrow & \lambda x\, [x + [\![\, \text{‘}\, \hat{}\, x + 3 \,\text{’} \,]\!]\,] \\
\equiv & \lambda x\, [x + [\![\, \text{‘}\, \hat{}(x + 3) \,\text{’} \,]\!]\,] \\
\rightarrow & \lambda x\, [x + (x + 3)]
\end{array}
$$

$\lambda Y\,\texttt{[}\lambda X\,\texttt{[}\lambda x\,\texttt{[}x + [\![X\,\texttt{(}x\ \texttt{=}\ x\texttt{)}\,\texttt{(}y\ \texttt{=}\ 3\texttt{)}]\!]\texttt{]]}\,\texttt{(`}x + [\![\hat{}\,Y\,\texttt{(}x\ \texttt{=}\ x\texttt{)}\,\texttt{(}y\ \texttt{=}\ y\texttt{)}]\!]\texttt{')]}$

$\texttt{`}y \times x\texttt{'}$