Modality with Explicit States of Knowledge

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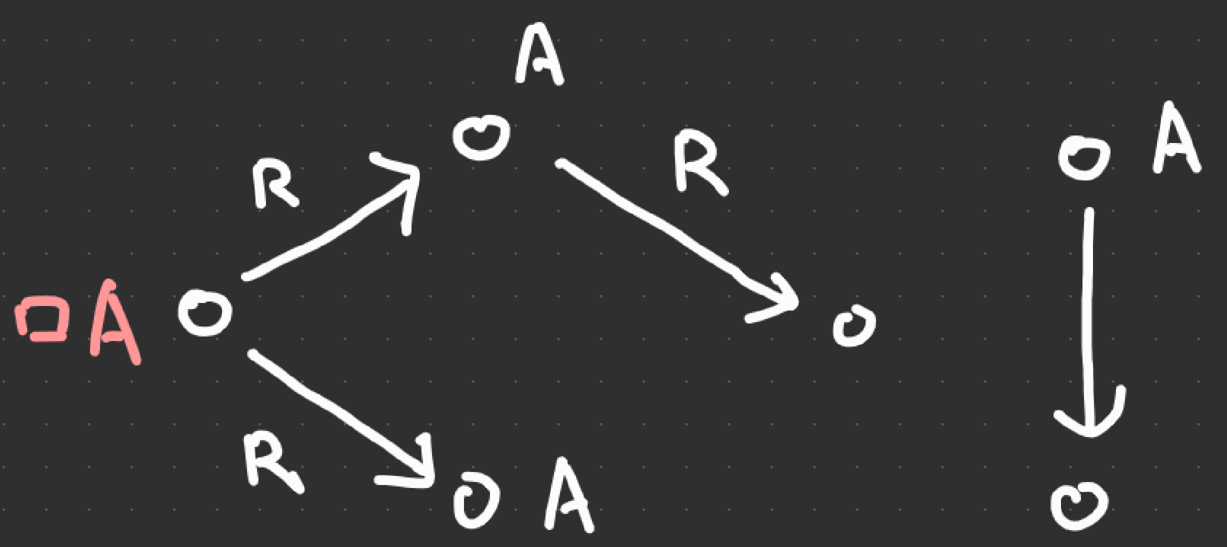
Background : Modal Logic

$\Box A$: A always holds

$$\mathcal{M} = \langle \underset{\text{worlds}}{W}, \underset{W \times W}{R}, \underset{W \times PVar \to Bool}{V} \rangle$$

with $R \subseteq W \times W$

$$\mathcal{M} \vDash_{\omega} \Box A \overset{\Delta}{\iff} \forall \underset{\in W}{\omega'} \text{ s.t. } \omega R \omega'. \ \mathcal{M} \vDash_{\omega'} A$$



Background: Intuitionistic Logic

$$\mathcal{M} = \langle W, \underset{}{\overset{\overset{W\times W}{\cap}}{\preceq}}, V \rangle$$

$$\mathcal{M} \models_\omega A \to B \overset{\Delta}{\iff} \forall \underset{\in W}{\omega'} \text{ s.t. } \omega \preceq \omega'.\ \mathcal{M} \models_{\omega'} A \Rightarrow \mathcal{M} \models_{\omega'} B$$

Informally, $\mathcal{M}$ captures the process of growing knowledge

- world ω $\mapsto$ state of knowledge
- $\omega_1 \preceq \omega_2$ $\mapsto$ ω_2 has more knowledge than ω_1
- Monotonicity: $\mathcal{M} \models_{\omega_1} A$ and $\omega_1 \preceq \omega_2 \Rightarrow \mathcal{M} \models_{\omega_2} A$

Bounded Modal Logic for Reasoning Monotonic Knowledge

Syntactic obj. for knowledge state

(reserve the classifier ! for empty knowledge)

Classifires $\gamma, \delta, \ldots$

Propositions $A, B ::= p \mid A \to B \mid \Box^{\gamma} A \mid \forall \gamma_1 \mathrel{\unrhd} \gamma_2 . A$

Bounded Modality

Universal Quantifiers for Classifiers

Kripke Semantics for Bounded Modal Logic

Model

$$\mathcal{M} = \langle \underset{\text{worlds}}{W}, \overset{\text{modal rel.}}{R}, \underset{\text{int. rel.}}{\preceq}, \overset{\text{valuation}}{V}, \underset{\text{world for empty knowledge}}{\omega^!} \rangle$$

(with several side-conditions)

World Assignment ← Classifires are interpreted as worlds

$$\rho ::= ! \mapsto \omega^! \mid \rho, \gamma \mapsto \omega$$

$$\boxed{\mathcal{M} \models^{\rho}_{\omega} A}$$

$$\mathcal{M} \models^{\rho}_{\omega} \Box^{\gamma} A \overset{\Delta}{\iff} \forall \omega' \in W \text{ s.t. } \omega R \omega'.\ \underbrace{\rho(\gamma)}_{\text{lower bound for } \omega'} \preceq \omega' \Rightarrow \mathcal{M} \models^{\rho}_{\omega'} A$$

$$\mathcal{M} \models^{\rho}_{\omega} \underbrace{\forall \gamma_1 :\succeq \gamma_2}_{\forall \gamma_1 \text{ s.t. } \gamma_2 \preceq \gamma_1} . A \overset{\Delta}{\iff} \forall \omega' \in W \text{ s.t. } \rho(\gamma_2) \preceq \omega'.\ \mathcal{M} \models^{\rho, \gamma_1 \mapsto \omega'}_{\omega} A$$

Bounded Modality

$\Box^{\gamma} A$ "A is valid assuming the knowledge γ"

Example:

"ALC is taking place in Kyoto" is known at γ

A = "There are many logicians in Kyoto"

$\Rightarrow$ $\Box^{\gamma} A$ is true ☺ $\Box^{!} A$ is false ☹

Universal Quantification over Classifiers

$\forall \gamma_1 :\geq \gamma_2 . A$ "A holds with any γ_1 that has grown from γ_2"

Example:

"ALC is taking place in Kyoto" is known at γ

B = "It is raining in Kyoto"

C = "Many logicians hold umbrellas in Kyoto"

$\Box^{\gamma} B \to \Box^{\gamma} C$ is true

$\forall \gamma' :\geq \gamma . (\Box^{\gamma'} B \to \Box^{\gamma'} C)$ is true $\Leftarrow$ stronger

Example Tautologies

Monotonicity $\Rightarrow$ $\forall \gamma_1 :\succeq !\,.\ \forall \gamma_2 :\succeq \gamma_1 .\ (\square^{\gamma_1} A \to \square^{\gamma_2} A)$

$$\forall \gamma_1 :\succeq !\,.\ (\forall \gamma_2 :\succeq \gamma_1 .\ \square^{\gamma_2} A) \leftrightarrow \square^{\gamma_1} A$$

$$\forall \gamma_1 :\succeq !\,.\ (\forall \gamma_2 :\succeq \gamma_1 .\ \square^{\gamma_2} A \to \square^{\gamma_2} B) \to \square^{\gamma_1} (A \to B)$$

if R is reflexive
$$\square^{!} A \to A$$
$$\forall \gamma_1 :\succeq !\,.\ \forall \gamma_2 :\succeq \gamma_1 .\ \square^{\gamma_2} \square^{\gamma_1} A \to \square^{\gamma_2} A$$

if R is transitive
$$\forall \gamma :\succeq !\,.\ \square^{\gamma} A \to \square^{!} \square^{\gamma} A$$

Motivation: Elaborating Scope-Safe Metaprogramming

(K+ 2016)

	In Logic,	In Prog. Lang.
γ	Knowledge State	$\Longleftrightarrow$ Variable Scope ← in λ-calculi
$\Box A$	A is valid	Code of A-expr (D&P 2001, D 2017)
$\Box^{\gamma} A$	A is valid assuming γ	Code of A-expr under the scope γ
$\forall \gamma_1 :\geq \gamma_2 . A$	Knowledge State Quantification	Polymorphism over Variable Scopes

Current Progress

- ✓ Kripke Semantics
- ✓ Constructive Natural-Deduction Proof System
  - Modal λ-calculus as term-assignment
- WIP Soundness & Completeness Proof

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- WIP Embedding LTL by K-BML

  $\uparrow$

  $K + K^{-1}((\Box A \to \Box B) \to \Box(A \to B))$
- WIP Strong Normalization via Reducibility

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Related Work

- Hybrid Logic (P 1967, B 2011)

  $A @ \alpha$ = "A holds at α"

  └ a nominal represents a world

  ⇒ similarity to classifiers in BML?

- Contextual Modal Logic (N+ 2008, M+ 2023)

  $[\vec{A} \vdash B]$ = "B is valid assuming all of $\vec{A}$"

  - Similar motivation to BML (Application to PL)

Let us know if you know relevant work

References (1/2)

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[D 2017] Davies, R. (2017). A Temporal Logic Approach to Binding-Time Analysis. *Journal of the ACM*, *64*(1), 1:1-1:45.
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[P 1967] Prior, A.N., 1967. *Past, Present and Future*, Oxford: Clarendon Press.

[B 2011] Braüner, T. (2011). *Hybrid Logic and its Proof-Theory* (Vol. 37). Springer Netherlands.

[N+ 2008] Nanevski, A., Pfenning, F., & Pientka, B. (2008). Contextual modal type theory. *ACM Transactions on Computational Logic*, *9*(3), 23:1-23:49.

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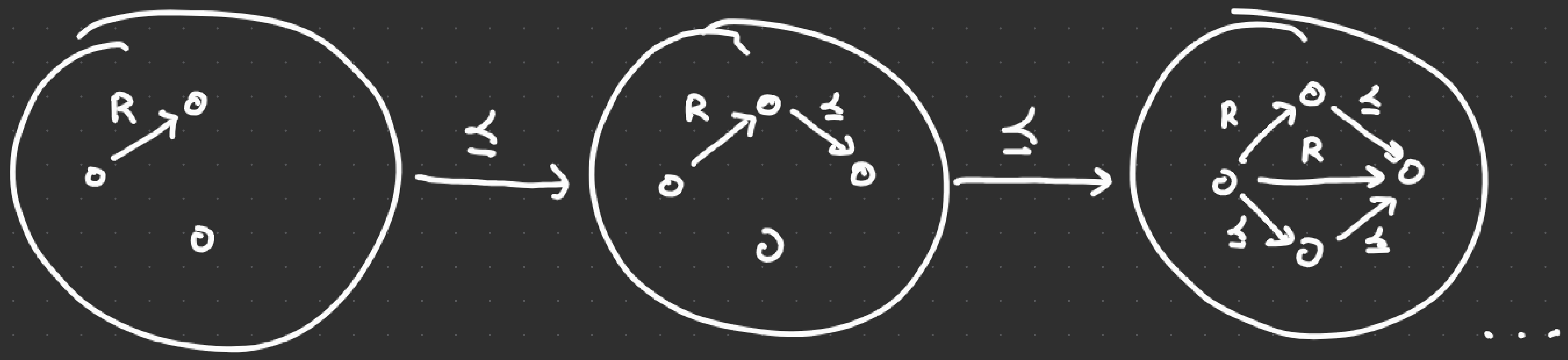
SUPPLEMENTARY MATERIALS

Intuitionistic Reasoning of BML

Int. relations appear in both layers

$$\mathcal{M} = \langle W, \preceq, \{D_\omega, R_\omega, \preceq_\omega, V_\omega\}_{\omega\in W}\rangle$$

BML model for each ω



Corresponds to modal λ-calculus

Motivation: Elaborating Scope-Safe Metaprogramming

(K+ 2016)

The Curry-Howard Isomorphism

	Logic	Programming Lang.
$\Box A$	A is valid	Code of A-expr (D&P 2001, D 2017)
γ	Knowledge state	Variable scope
$\Box^{\gamma} A$	A is valid assuming γ	Code of A-expr under the scope γ
$\forall \gamma_1 :\geq \gamma_2 . A$	Knowledge state quantification	Polymorphism over variable scopes