Effective Limit in Computable Analysis

Takeuti Izumi

Faculty of Sciences, Toho University, Miyama, Hunabasi, Tiba, 274-8510, JAPAN

takeuti@is.sci.toho-u.ac.jp

Abstract: This paper introduce the notion of effective limit, which gives an alternative definition of effective uniform topology. The aim of this paper is to combine the precedent works of algebraic structure and of limit structure, which have appeared in studies of computable analysis. This paper also compares the definition of computable functions based on representations and that based on computability structure, and shows that both of them are essentially equivalent to each other.

Keywords: effective limit, uniform topology, computability structure, representation

1 Introduction

Most of the studies of computable analysis have been capturing the computability of structures only over real numbers. However, there are started some studies on other general mathematical structures recently.

Hertling studied computability over algebraic structures [H], however he did not discuss topology. Schröder studied week limit spaces [S], which are generalisation of topological spaces, however he did not discuss modulus of convergence. The author believes that the notions of topology and modulus of convergence are most important in the study of computable analysis. This work aims to combine their works, and to apply to the discussion on modulus of convergence.

In discussing modulus of convergence, it seems natural to do it with uniform topology, which Yasugi et al. studied [Y]. They discussed the computability under uniform topology with computability structure, while Hertling and Schröder did it with representation.

There are several methods in studying computable analysis. One of them is representation and another is computability structure.

The notion of representation has been studied by Weihrauch [W] or others. It is a kind of a mechanical method. A representation r of a mathematical structure X is a surjective partial function of Σ^ω into X. A function $f : X \to X$ is computable if the is a partial recursive functional $\bar{f}$ of Σ^ω into Σ^ω such that $f \circ r = r \circ \bar{f}$. We will use $\mathbb{N} \to \mathbb{N}$ as the domain of representation Σ^ω, while some works use $\{0, 1\}^\omega$ as Σ^ω. As the author's opinion, it seems that representations have too much information which comes from the details of implementation of computation models.

The notion of computability structure is studied by Pour-El et al. [P]. It is a kind of an axiomatic method. A computability structure over a mathematical structure X is a subset of X^ω which satisfies the axioms of computability structure. A function f over X is called computable if f is locally uniformly continuous and preserves the computability structure over X. In order to assert that a computability structure is natural, the mathematical structure X has been requested to have a distance. Yasugi attempts to replace the distance with effective uniform topology [Y].

This work aims to compare these two ways, that by computability structure and that by representation. The author would like to show that the notion of computability defined by each way is essentially equivalent to each other.

There are several ways to define effective uniform topology. Yasugi et al. defined it with effective uniform neighbourhood system [Y]. This work defines effective uniform topology topology with effective limit, which has not been used in any other works. An effective limit is a partial function of arity ω over the underlying set, while an effective neighbourhood system is a family of subsets. It seems that a partial function is more familiar in computational theory than a family of subset is.

There are three similar words *'effective'*, *'recursive'*, and *'computable'*. All of them appear in this paper. There are only little difference in the senses of these words. In this paper, the word 'recursive' is used only for the notions concerned to recursive functions over natural numbers. The word 'computable' is used only for the notion of computability structure which defined by Pour-El et al. [P]. The word 'effective' is used for other mathematical structures. Thus, a function represented by a recursive Type-2 function is called a recursive function in this paper, although Weihrauch or others call it a computable function [W].

2 Uniform Topology

Notation 2.1 We write $\mathbb{N}$ for $\{1, 2, 3, ...\}$.

Definition 2.2 (Partial function) A function f is a partial function of X into Y iff f is a function of $X' \subset X$ into $Y' \subset Y$. This X' is called the *domain* of f and written as $dom(f)$. We write $f : X \to_p Y$ when f is a partial function of X into Y. For $x \in X$, we say that $f(x)$ is *defined* when $x \in dom(f)$. When we write $f(x) = y$ or something, we implicitly assert $x \in dom(f)$.

The *range* of $f : X \to_p Y$ is the set $\{f(x) \in Y \mid x \in dom(f)\}$ and written as *range*(f). For $f : X \to_p Y$ and $g : Y \to_p Z$, the concatenation $g \circ f : X \to_p Z$ is defined as following: $dom(g \circ f) = \{x \in X \mid x \in dom(f), f(x) \in dom(g)\}$ and $g \circ f(x) = g(f(x))$ for $x \in dom(g \circ f)$.

Notation 2.3 We write $X \to Y$ for the function space $\{f \mid f : X \to Y\}$, and $X \to_p Y$ for the set of all the partial functions $\{f \mid f : X \to_p Y\}$.

Definition 2.4 (Modular limit) Let X be a set. A partial function LM is a *modular limit* over X if there is a strictly increasing function $m : \mathbb{N} \to \mathbb{N}$ and the followings hold:

0. $LM : X^\omega \to_p X$.

1. For each $x \in X$, $LM(x, x, x, ...) = x$.
2. Let f be a function of $\mathbb{N} \to \mathbb{N}$ such that $f(n) \geq n$ for any $n \in \mathbb{N}$. If $(x_1, x_2, x_3, ...) \in dom(LM)$, then $(x_{f(1)}, x_{f(2)}, x_{f(3)}, ...) \in dom(LM)$ and $LM(x_{f(1)}, x_{f(2)}, x_{f(3)}, ...) = LM(x_1, x_2, x_3, ...)$.
3. Let $\{x_{i,j}\}_{i,j}$ be a double sequence in X. Let $(y_1, y_2, y_3, ...)$ be a sequence in X. and z be a point in X. Suppose that $y_i = LM(x_{i,1}, x_{i,2}, x_{i,3}, ...)$ for each i.

   If $LM(y_1, y_2, y_3, ...) = z$, then $LM(x_{m(1),m(1)}, x_{m(2),m(2)}, x_{m(3),m(3)}, ...) = z$.
4. Let $\{x_{i,j}\}_{i,j}$ be a double sequence in X, that is, $x_{i,j} \in X$ for any $i, j \in \mathbb{N}$. Let $(y_1, y_2, y_3, ...)$ be a sequence in X. and z be a point in X. Suppose that $y_i = LM(x_{i,1}, x_{i,2}, x_{i,3}, ...)$ for each i.

   If $LM(x_{1,1}, x_{2,2}, x_{3,3}, ...) = z$, then $LM(y_{m(1)}, y_{m(2)}, y_{m(3)}, ...) = z$.

The function $m(\,)$, which appears in the conditions 3 and 4, is called a *modulus of diagonal convergence.*

Definition 2.5 (Effective limit) A modular limit is called an *effective limit* when the modulus of diagonal convergence of it is a recursive function.

Remark 2.6 The conditions 3 and 4 in the definition of modular limit means that all the converging sequence under the modular limit are uniformly converging.

Remark 2.7 A modular limit LM over a set X induces topology over X, where the closure operator $\mathcal{F}$ and the internal kernel operator $\mathcal{G}$ are defined as follows:

$$\begin{aligned}
\mathcal{F}(E) &= \{x \in X \mid \exists (y_1, y_2, y_3, ...) \in E^\omega.\ LM(y_1, y_2, y_3, ...) = x\}, \\
\mathcal{G}(E) &= \{x \in X \mid \forall (y_1, y_2, ...) \in E^\omega.\ LM(y_1, y_2, ...) = x \Rightarrow \\
&\qquad\qquad\qquad\qquad\qquad\qquad \exists n.\forall i \geq n.\ y_i \in E\} \\
&= X - \mathcal{F}(X - E).
\end{aligned}$$

It is easy to seen that the topological space X with this topology is always a Hausdorff space.

Remark 2.8 There are several well-known definitions of uniform topology which are equivalent to each other. One of them is the definition by uniform neighbourhood systems. The following several lemmata show that modulus limit also gives the equivalent definition of uniform topology.

Definition 2.9 (Uniform neighbourhood system) Let X be a set. The series of subsets $\{V_i(x)\}_{i \in \mathbb{N}, x \in X}$ is a *uniform neighbourhood system* iff there are functions $m, m' : \mathbb{N} \to \mathbb{N}$ and the followings hold:

1. $x \in V_i(x) \subset X$ for each $i \in \mathbb{N}$ and each $x \in X$.

2. $V_i(x) \supset V_j(x)$ for $i < j$.

3. There is a function $m : \mathbb{N} \to \mathbb{N}$ such that $y \in V_{m(i)}(x)$ implies $x \in V_i(y)$.

4. There is a function $m' : \mathbb{N} \to \mathbb{N}$ such that $y \in V_{m'(i)}(x)$ implies $V_{m'(i)}(y) \subset V_i(x)$.

The functions m and m' in the conditions 3 and 4 are called the *moduli*.

Definition 2.10 (Effective uniform neighbourhood system) The uniform neighbourhood system is *effective* if both of the moduli are recursive functions.

Proposition 2.11 *Let X be a set and $\{V_i(x)\}_{i,x}$ be a uniform neighbourhood system over X. Then, X is a Hausdorff space iff $\bigcap_{i \in \mathbb{N}} V_i(x) = \{x\}$ for any $x \in X$.*

Definition 2.12 (Closeness relation) Let LM be the modular limit of X. The relation $x \stackrel{n}{\to} y$ holds iff there is a sequence $(x_1, x_2, x_3, ...)$ such that $x = x_i$ and $LM(x_1, x_2, x_3, ...) = y$. We call this relation *closeness relation*.

Proposition 2.13 *The following conditions hold. In the conditions, the function $m(\ )$ is the modulus of diagonal convergence of the modular limit in the followings.*

1. *If $i \leq j$ and $x \stackrel{j}{\to} y$, then $x \stackrel{i}{\to} y$.*

2. *If $m(i) \leq j$ and $x \stackrel{j}{\to} y$, then $y \stackrel{i}{\to} x$.*

3. *If $m(i) \leq j$, $x \stackrel{j}{\to} y$ and $y \stackrel{j}{\to} z$, then $x \stackrel{i}{\to} z$.*

4. *If $m(m(i)) \leq j$ and $x \stackrel{j}{\to} z \stackrel{j}{\leftarrow} y$, then $x \stackrel{i}{\to} y$.*

5. *If it holds that $x_i \stackrel{m(i)}{\to} y$ for each i, then $LM(x_1, x_2, x_3, ...) = y$.*

Lemma 2.14 *Let X be a set and LM be a modular limit of modulus m over X. For each $i \in \mathbb{N}$ and each $x \in X$, a subet $V_i(x) \subset X$ is defined as $V_i(x) = \{y \mid y \stackrel{i}{\to} x\}$ Then, $\{V_i(x)\}_{i,x}$ is a uniform neighbourhood system of modulus m over X, and it makes X a Hausdorff space.*

Moreover, if LM is an effective limit, then $\{V_i(x)\}_{i,x}$ is an effective uniform neighbourhood system.

Lemma 2.15 *Let X be a set and $\{V_i(x)\}_{i,x}$ be a uniform neighbourhood system over X which makes X a Hausdorff space. A partial function $LM : X^\omega \to_p X$ is defined as follows:*

$$dom(LM) = \{(x_1, x_2, x_3, ...) \mid \exists y \in X.\ \forall i \in \mathbb{N}.\ x_i \in V_i(y)\},$$

$$LM(x_1, x_2, x_3, ...) = y \text{ iff } \forall i \in \mathbb{N}.\ x_i \in V_i(y).$$

Note that such x is uniquely determined, because X is a Hausdorff space by the neighbourhood system $\{V_i(x)\}_{i,x}$.

Then, LM is a modular limit of modulus m over X.

Moreover, if $\{V_i(x)\}_{i,x}$ is an effective uniform neighbourhood system, then LM is an effective limit.

Remark 2.16 The previous two lemmata (Lem. 2.14 & 2.15) mean that modular limit is corresponds to uniform neighbourhood system, and effective limit is corresponds to effective uniform neighbourhood system.

Example 2.17 Let $\mathbb{R}$ be the set of all the real numbers. Put a partial function $LimE : \mathbb{N}^\omega \to_p \mathbb{N}$ as:

$$dom(LimE) = \left\{ (x_1, x_2, x_3, ...) \,\middle|\, |x_i - x_j| < \frac{1}{i} + \frac{1}{j} \right\},$$

$$LimE(x_1, x_2, x_3, ...) = x \text{ iff } \lim_{i \to \infty} x_i = x.$$

Then, this partial function $LimE$ is an effective limit with the modulus of diagonal convergence $n \mapsto 2n$.

For $i \in \mathbb{N}$ and $x \in \mathbb{R}$, put $V_i^{\mathbb{R}}(x) \subset \mathbb{R}$ as $y \in V_i^{\mathbb{R}}(x)$ iff $|y - x| \leq 1/i$. Then, $V_i^{\mathbb{R}}(x) = \{y \mid y \stackrel{n}{\to} x\}$ and $\{V_i^{\mathbb{R}}(x)\}_{i,x}$ is an effective uniform neighbourhood system.

Example 2.18 The functional space $\mathbb{N} \to \mathbb{N}$ is regarded as a topological space. Define a partial function $LM_{\mathbb{N},\mathbb{N}} : (\mathbb{N} \to \mathbb{N})^\omega \to_p \mathbb{N} \to \mathbb{N}$ as:

$$(x_1, x_2, x_3, ...) \in dom(LM_{\mathbb{N},\mathbb{N}}) \text{ iff for any } i \leq j < k,\ x_j(i) = x_k(i),$$

$$\text{For } (x_1, x_2, x_3, ...) \in dom(LM_{\mathbb{N},\mathbb{N}}),\ LM_{\mathbb{N},\mathbb{N}}(x_1, x_2, x_3, ...) = (i \mapsto x_i(i)).$$

The condition 1 means that x_j and x_k have the common initial segment of length j where $j < k$. Then, this $LM_{\mathbb{N},\mathbb{N}}$ is a modular limit with the identity function $i \mapsto i$ as the modulus of diagonal convergence, and it induces the ordinary topology of $\mathbb{N} \to \mathbb{N}$.

3 Algebraic Structure

Definition 3.1 (Algebraic structure) A sequence $X = (|X|, f_1, f_2, f_3, ..., R_1, R_2, R_3, ...)$ is an *algebraic structure* iff it consists of the underlying set $|X|$, partial functions $f_i : |X|^{arity(f_i)} \to_p |X|$, and subsets $R_i \subset |X|^{arity(R_i)}$. An algebraic structure X has finite partial functions or countably many infinite partial functions, and also X has finite relations or countably many infinite relations. For any f_i and any R_i, the arities $arity(f_i)$ and $arity(R_i)$ belong to $\{0\} \cup \mathbb{N} \cup \{\omega\}$. The set of the functions and relations $\{f_1, f_2, f_3, ..., R_1, R_2, R_3, ...\}$ are called the signature of X.

We sometimes identify X to $|X|$, and simply write X for $|X|$.

We may define that an algebraic structure has many sorted, or it refers some other algebraic structures.

Definition 3.2 (Algebraic structure with uniform topology) Let $X = (|X|, f_1, f_2, f_3, ..., R_1, R_2, R_3, ...)$ be an algebraic structure. Suppose that a partial function $f_i = LM$ belongs to the signature, and $f_i = LM$ satisfies the definition of modular limit over the domain $|X|$. Then, this X is an *algebraic structure with uniform topology* by the modular limit LM.

If the modular limit is effective, then it is called an *algebraic structure with effective uniform topology*

Example 3.3 we define an algebraic structure $\mathbb{R} = (|\mathbb{R}|, 0, 1, +, -, \times, /, LimE, <)$ as follows.

The set $|\mathbb{R}|$ is the set of all the real numbers. The 0-ary functions 0 and 1 are the numbers 0 and 1 themselves. The binary functions $+$, $-$ and $\times$ are the ordinary summation, subtraction and multiplication. The partial binary function $/$ is the ordinary division. The partial function $LimE : \mathbb{R}^\omega \to_p \mathbb{R}$ is defined as in Example 2.17. The relation $<$ is the ordinary inequality without equality.

Then, This $\mathbb{R}$ is an algebraic structure with effective uniform topology. We sometimes identify $\mathbb{R}$ to $|\mathbb{R}|$.

4 Representation

Notation 4.1 We identify a function $f : \mathbb{N} \to \mathbb{N}$ to a infinite sequence $(f(0), f(1), f(2), ...) \in \mathbb{N}^\omega$. For a finite sequence $x \in \mathbb{N}^*$ and a finite or infinite sequence $y \in \mathbb{N}^* \cup \mathbb{N}^\omega$, we write $x \sqsubseteq y$ when x is an initial segment of y.

Definition 4.2 (Recursive Type-2 function) A partial function of $\mathbb{N}^\omega \to_p \mathbb{N}^\omega$ is called a *partial Type-2 function* or *partial functional.* A partial functional $F : \mathbb{N}^\omega \to_p \mathbb{N}^\omega$ is *recursive* iff there is a partial recursive function $f : \mathbb{N}^* \to_p \mathbb{N}$ which satisfies the following three conditions.

1. The function f is monotone with respect to $\sqsubseteq$, that is, for any $y \sqsubseteq z \in \mathbb{N}^*$, $f(y) \sqsubseteq f(z)$.
2. For each $x \in \mathbb{N}^\omega$, $x \in dom(F)$ iff there are arbitrary long $f(y)$ such that $y \sqsubseteq x$, that is, for every $n \in \mathbb{N}$, there exists $y \sqsubseteq x$ such that $f(y)$ is longer than the length n.
3. For $x \in dom(F)$, $F(x)$ is the infinite sequence such that $f(y) \sqsubseteq F(x)$ for any $y \sqsubseteq x$.

Remark 4.3 This definition is equivalent to the notion of recursive functions relative to another functions, which appears in the literature [O]. And also this is equivalent to the notion defined by Type-2 machines in the literature [W].

Definition 4.4 (Representation) A partial function $r : \mathbb{N}^\omega \to_p X$ is a *representation* of X iff it is a surjection, that is, $range(r) = X$.

Definition 4.5 (Pairing) A *pairing function* $\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a standard bijection of $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, which is defined as:

$$\langle m, n \rangle = \frac{(m+n-1)(m+n-2)}{2} + m,$$

Remark 4.6 With this paring function $\langle -, - \rangle$, we can regard a unary function as a binary function, such as a unary function $(n \mapsto f(n)) : \mathbb{N} \to \mathbb{N}$ as a binary function $((m, n) \mapsto f(\langle m, n \rangle)) : \mathbb{N}^2 \to \mathbb{N}$.

Definition 4.7 (Recursiveness with respect to representations) Let X and Y be sets. Let r and r' be representations of X and of Y respectively. A partial function $f : X \to_p Y$ is *recursive* with respect to r and r' iff there is a Type-2 partial recursive functional $F : \mathbb{N}^\omega \to_p \mathbb{N}^\omega$ such that $f \circ r) = r' \circ F$. If $X = Y$ and $r = r'$, then we say simply that the f is recursive with respect to r.

Remark 4.8 We say that $f : X^2 \to_p X$ is recursive with respect to r iff it is recursive with respect to (r', r) where r' is defined as:

$r'(n \mapsto \langle w_1(n), w_2(n) \rangle) = (r(w_1), r(w_2))$ for any $w_1, w_2 \in \mathbb{N}^\omega$.

It is similarly for $f : X^3 \to_p X$, $f : X^4 \to_p X$ and so forth.

We say that $f : X^\omega \to_p X$ is recursive with respect to r iff it is recursive with respect to (r'', r) where r'' is defined as:

$r''(w)(n) = r(i \mapsto w(\langle n, i \rangle))$ for any $w \in \mathbb{N}^\omega$.

Definition 4.9 (Reducibility and continuous reducibility) Let X be a set, and r be a representation of X. A partial function $f : \mathbb{N}^\omega \to_p X$ is *continuously reducible* to r if there is a partial continuous functional $F : \mathbb{N}^\omega \to \mathbb{N}^\omega$ such that $f = r \circ F$. The partial function f is *reducible* to r if such F is recursive.

Definition 4.10 (Final continuity) Let X be a topological space. A representation $r : \mathbb{N}^\omega \to_p X$ is *finally continuous* if r is continuous and every partial continuous function of $\mathbb{N}^\omega \to_p X$ is continuously reducible to r.

Remark 4.11 The notion of final continuity defined above is called admissibility by Weihrauch [W] or others. This notion is just a topological notion, and does not include none of effectivity.

Lemma 4.12 *Let X be an algebraic structure with uniform topology, and LM be the modular limit of X, Let r be a representation of X. Suppose that LM is computable with respect to r. Then any continuous partial function $f : \mathbb{N}^\omega \to X$ r is continuously reducible to r.*

Outline of the proof. There are a function $a : \mathbb{N} \to \mathbb{N}^\omega$ and a continuous map $e : dom(f) \to \mathbb{N}^\omega$ such that $LM(f \circ a \circ (e(w))) = f(w)$ for any $w \in dom(f)$. There is a function $b : \mathbb{N} \to \mathbb{N}^\omega$ such that $r(\underline{b(i)}) = f(a(i))$ for each $i \in \mathbb{N}$. Moreover, there is a partial recursive functional $\overline{LM}$ such that $r(\overline{LM}(w)) = LM(r \circ w)$ for any $w \in \mathbb{N}^\omega$. Therefore, we have

$$f(w) = LM(f \circ a \circ (e(w))) = LM(r \circ b \circ (e(w))) = r(\overline{LM}(b \circ (e(w))))$$

and the map $w \mapsto \overline{LM}(b \circ (e(w)))$ is continuous.

Corollary 4.13 *If a representation r of X is continuous, and the modular limit of X is recursive with respect to r, then r is finally continuous.*

5 Computability Structure

Remark 5.1 A binary partial recursive function $f : \mathbb{N} \times \mathbb{N} \to_p \mathbb{N}$ induces a total function of natural numbers into partial recursive functions $(m \mapsto (n \mapsto f(m,n))) : \mathbb{N} \to (\mathbb{N} \to_p \mathbb{N})$. When we say a recursive function $g : \mathbb{N} \to (\mathbb{N} \to_p \mathbb{N})$, we will denote that there is a binary partial recursive function $f : \mathbb{N} \times \mathbb{N} \to_p \mathbb{N}$ such that $g : m \mapsto g(m) = (n \mapsto f(m,n))$.

Definition 5.2 (Term) Let $\{f_1, f_2, ..., R_1, R_2, ...\}$ be a signature and $V = \{v_1, v_2, v_3, ...\}$ be a set of countably infinitely many variables. Then we define the set of formal expressions $Term_\alpha$ which is defined for each ordinal α as follows:

$$\begin{aligned} Term_0 &= V, \\ Term_{\alpha+1} &= Term_\alpha \\ &\qquad \cup \{f_i\langle t_1, t_2, t_3, ..., t_{arity(f_i)}\rangle \mid arity(f_i) \in \{0\} \cup \mathbb{N}, t_j \in Term_\alpha\} \\ &\qquad \cup \{f_i\langle t_1, t_2, t_3, ...\rangle \mid arity(f_i) = \omega, t_j \in Term_\alpha\}, \\ Term_\alpha &= \bigcup_{\beta<\alpha} Term_\beta \text{ for a limit ordinal } \alpha. \end{aligned}$$

It is easy to seen that $Term_\alpha$ is saturated for uncountable ordinals α, because the arity of every function is countable. Thus $Term_\alpha$ is unique for any uncountable ordinal α. We define $Term$ as:

$$Term = Term_\alpha \text{ for the least uncountable ordinal } \alpha.$$

Elements of $Term$ are called *terms*.

Definition 5.3 (Evaluation) Let X be a structure. We define a partial function $[\![-]\!] : Term \times (\mathbb{N} \to_p X) \to_p X$ such as:

- For a variable $v_i \in V = Term_0$ and a partial function $\rho : \mathbb{N} \to_p X$,

  $[\![v_i]\!]_\rho$ is defined iff so is $\rho(i)$, and $[\![v_i]\!]_\rho = \rho(i)$.

- For a term $f_i\langle t_1, t_2, ...\rangle \in Term_{\alpha+1} - Term_\alpha$ and a partial function $\rho : \mathbb{N} \to_p X$,

  $[\![f_i\langle t_1, t_2, ...\rangle]\!]_\rho$ is defined iff $f_i([\![t_1]\!]_\rho, [\![t_2]\!]_\rho, ...)$ is defined

  and $[\![f_i\langle t_1, t_2, ...\rangle]\!]_\rho = f_i([\![t_1]\!]_\rho, [\![t_2]\!]_\rho, ...)$.

Definition 5.4 (Standard representation of terms) We define a representation r_T of $Term$ with representations r_α of $Term_\alpha$ for any ordinals α.

Let $j_i : \mathbb{N} \to \mathbb{N}$ be a function such that $j_i(n) = 1 + \langle i, n \rangle$. Note that $\bigcup_{i \in \mathbb{N}} j_i(\mathbb{N}) = \{2, 3, 4, ...\}$ and all the summands are disjoint to each other.

The function r_0 of $V = Term_0$ is defined as follows.

- For $x \in \mathbb{N}^\omega$, $x \in dom(r_0)$ iff $x(1) = 1$.
- For $x \in dom(r_0)$, $r_0(x) = v_{x(2)}$.

The representation $r_{\alpha+1}$ of $Term_{\alpha+1}$ is defined as follows.

- For $x \in \mathbb{N}^\omega$, $x \in dom(r_0)$ iff either

  $x \in dom(r_\alpha)$

  or,

  $x(1)$ is defined, $x(1) \neq 1$, the signature has the function $f_{x(1)-1}$

  and $x \circ j_i \in dom(r_\alpha)$ for each $i < arity(f_{x(1)-1})$.

- For $x \in dom(r_{\alpha+1})$,

  if $x \in dom(r_\alpha)$ then $r_{\alpha+1}(x) = r_\alpha(x)$,

  otherwise $r_{\alpha+1}(x) = f_{x(1)-1}\langle r_\alpha(x \circ j_1), r_\alpha(x \circ j_2), r_\alpha(x \circ j_3), ...\rangle$.

The representation r_α of $Term_\alpha$ for a limit ordinal α is defined as follows.

- $dom(r_\alpha) = \bigcup_{\beta < \alpha} dom(r_\beta)$.
- For $x \in dom(r_\alpha)$,

  if $x \in dom(r_0)$ then $r_\alpha(x) = r_0(x) = v_{x(1)}$,

  and if $x \in dom(r_{\beta+1}) - dom(r_\beta)$ then $r_\alpha(x) = r_{\beta+1}(x)$.

This r_α is saturate as α is uncountable. We define r_T as $r_T(x) = r_\alpha(x)$ for the least uncountable ordinal α.

It is obvious that this $r_\alpha : \mathbb{N}^\omega \to_p Term_\alpha$ is surjective. Thus, r_T is a surjection into $Term$.

Definition 5.5 (Computability structure) Let X be an algebraic structure. A subset $S \subset \mathbb{N} \to X$ is a *computability structure* over X if S satisfies the followings:

1. (Permutation) For each $s \in S$ and each total recursive function $f : \mathbb{N} \to \mathbb{N}$, it holds that $s \circ f \in S$.
2. (Merging) For any $s, s' \in S$, there is $s'' \in S$ such that $s''(2n - 1) = s(n)$ and $s''(2n) = s'(n)$.

3. (Effective sequence of terms) For each $s \in S$ and each recursive function $f : \mathbb{N} \to (\mathbb{N} \to_p \mathbb{N})$, if $f(m) \in dom(r_T)$ and $[\![r_T(f(m))]\!]_{(i \mapsto s(\langle n,i \rangle))}$ is defined for each $m, n \in \mathbb{N}$, then there is $s' \in S$ such that $s'(\langle m,n \rangle) = [\![r_T(f(m))]\!]_{(i \mapsto s(\langle n,i \rangle))}$.

Example 5.6 The algebraic structure $\mathbb{R}$ is defined as in Example 3.3. The standard computable structure over $\mathbb{R}$ is

$$\begin{aligned} S_{\mathbb{R}} = \{ (n \mapsto LimE(i \mapsto (-1)^{h(n,i)} g(n,i)/f(n,i))) \in \mathbb{R}^{\omega} \mid \\ & f,\ g \text{ and } h \text{ are total recursive functions.} \\ & \text{The value of } LimE \text{ is defined for each } n.\ \} \end{aligned}$$

which defined as by Pour-El et al [P].

Example 5.7 Regard $\mathbb{N}^{\omega}$ as an algebraic structure with uniform topology ($\mathbb{N}^{\omega}$, $LM_{\mathbb{N},\mathbb{N}}$), as in Example 2.18. Put $S_{\mathbb{N}^{\omega}}$ as

$$\{ f : \mathbb{N} \to \mathbb{N}^{\omega} \mid f(i)(j) = g(i,j),\ g \text{ is a binary total recursive function} \}.$$

Then, this $S_{\mathbb{N}^{\omega}}$ is a computability structure over $\mathbb{N}^{\omega}$. This $S_{\mathbb{N}^{\omega}}$ is the standard computability structure of it.

Definition 5.8 (Computable function) Let X and Y be algebraic structure with effective uniform topology, and LM^X and LM^Y be the effective limits of them, respectively. Let S_X and S_Y be computability structures of X and Y, respectively.

Then, a partial function $f : X \to Y$ is *computable* if there exists a total recursive function $m : \mathbb{N}^2 \to \mathbb{N}$ and the following five conditions hold:

1. There is $a \in S_X$ such that $s(\mathbb{N})$ is dense in $dom(f)$.
2. For each $s \in S_X$, there is a partial recursive function $i : \mathbb{N}^2 \to_p \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $s(n) \in dom(f)$, then $LM^X(a(i(n,1)), a(i(n,2)), a(i(n,3)), ...) = s(n)$.
3. $f \circ a \in S_Y$.
4. For every $(i_1, i_2, i_3, ...) \in \mathbb{N}^{\omega}$, if $(a(i_1), a(i_2), a(i_3), ...) = dom(LM^X)$, then there is a sequence $(j_1, j_2, j_3, ...) \in \mathbb{N}^{\omega}$ such that $m(i_{j_n}, n) \leq j_n$ for any $n \in \mathbb{N}$.
5. if $LM^X(a(i_1), a(i_2), a(i_3), ...) = x$, and $m(i_{j_n}, n) \leq j_n$ for any $n \in \mathbb{N}$, then $LM^Y(f(a(i_{j_1})), f(a(i_{j_2})), f(a(i_{j_3})), ...) = f(x)$.

The points of $a(\mathbb{N})$ is called *approximating points*, and m is called the *modulus*.

Proposition 5.9 *Let $f : X \to_p Y$ and $g : Y \to_p Z$ be computable functions with respect to (S_X, S_Y) and (S_Y, S_Z), respectively. If $range(f) \subset dom(g)$, then $g \circ f$ is computable with respect to (S_X, S_Z).*

Definition 5.10 (Induced computability structure) Let X be an algebraic structure and r be a representation of X such that all the signatures are recursive with respect to r. Then, a computability structure S_r is defined as follows:

> For each $s \in \mathbb{N} \to X$, $s \in S_r$ iff there is a total recursive function $f : \mathbb{N} \to \mathbb{N}^\omega$ such that $s = r \circ f$.

This S_r is called the *induced computability structure* of r.

Notation 5.11 We abbreviate the following condition over a quadruple (X, LM, r, S) as the condition $(*)$:

1. The first component X is an algebraic structures with effective uniform topology.
2. The second LM is the effective limits of X.
3. The third r is a representation of X such that all the partial functions of the signature of X are recursive with respect to r. Thus, S_r is a computable structures over X.
4. As for the forth component, $S = S_r$.

Lemma 5.12 *Let (X, LM, r, S_r) be a quadruple which satisfies the condition $(*)$. Then, for each function $f : \mathbb{N}^\omega \to X$, if f is computable with respect to $(S_{\mathbb{N}^\omega}, S_r)$, then f is reducible to r.*

Outline of the proof. The proof follows the similar way to the proof of Lemma 4.12. There are recursive functions $a, b : \mathbb{N} \to \mathbb{N}^\omega$ and partial recursive functionals $e : dom(f) \to \mathbb{N}^\omega$ and $\overline{LM} : (\mathbb{N}^\omega)^\omega \to_p \mathbb{N}^\omega$ such that

$$f(w) = LM(f \circ a \circ (e(w))) = LM(r \circ b \circ (e(w))) = r(\overline{LM}(b \circ (e(w))))$$

and the map $w \mapsto \overline{LM}(b \circ (e(w)))$ is recursive.

Corollary 5.13 *A partial function $f : \mathbb{N}^\omega \to_p \mathbb{N}^\omega$ is recursive if it is computable with respect to $S_{\mathbb{N}^\omega}$.*

Lemma 5.14 *Let f be a partial function of $\mathbb{N}^\omega \to_p \mathbb{N}^\omega$. Suppose that f is recursive and there is $a \in S_{\mathbb{N}^\omega}$ such that $a(\mathbb{N})$ is dense in $dom(f)$. Then, f is computable with respect to $S_{\mathbb{N}^\omega}$.*

Corollary 5.15 *Let f be a partial function of $\mathbb{N}^\omega \to_p \mathbb{N}^\omega$. Suppose that there is $a \in S_{\mathbb{N}^\omega}$ such that $a(\mathbb{N})$ is dense in $dom(f)$. Then, f is recursive iff it is computable with respect to $S_{\mathbb{N}^\omega}$.*

Lemma 5.16 *Let (X, LM^X, r, S_X) be a quadruple which satisfies the condition $(*)$. Let Y be an algebraic structure with effective uniform topology, and LM^Y be the effective limit of it. Let S_Y be a computability structure over Y. Suppose that r is computable with respect to $(S_{\mathbb{N}^\omega}, S_X)$.*

Then, for any $f : X \to_p Y$, if $f \circ r$ is computable with respect to $(S_{\mathbb{N}^\omega}, S_Y)$, then f is computable with respect to (S_X, S_Y).

Outline of the proof. It suffices to check each condition in Definition 5.8. As for the modulus, we can make the modulus $m : \mathbb{N}^2 \to \mathbb{N}$ effectively as below. Because $f \circ r$ is computable, for any $w \in \mathbb{N}^\omega$ and $n \in \mathbb{N}$, we can effectively find i_n such that if $y' \xrightarrow{n} y$ implies $r(y') \xrightarrow{i_n} r(y)$. Because r is finally continuous, for any $w \in \mathbb{N}^\omega$ and $n \in \mathbb{N}$, we can effectively find i_n such that if $w' \xrightarrow{n} w$ implies $r(w') \xrightarrow{i_n} r(w)$.

Theorem 5.17 (Main theorem) *Let (X, LM^X, r_X, S_X) and (Y, LM^Y, r_Y, S_Y) be quadruples which satisfy the condition $(*)$. Suppose that r_X and r_Y are computable with respect to $(S_{\mathbb{N}^\omega}, S_X)$ and $(S_{\mathbb{N}^\omega}, S_Y)$, respectively.*

Let $f : X \to_p Y$ and $F : \mathbb{N}^\omega \to_p \mathbb{N}^\omega$ be partial functions which satisfy the following:

1. $f \circ r_X = r_Y \circ F$
2. *range*$(F) \subset$ *dom*(r_Y)
3. *There is $a \in S_{\mathbb{N}^\omega}$ such that $a(\mathbb{N})$ is dense in dom(F).*

Then, F is recursive iff f is computable with respect to (S_X, S_Y).

Proof. It is derived form Lemmata 5.12 and 5.16. ∎

6 Computability in Yasugi's Sense

Remark 6.1 There is another definition of computable functions, which is given by Yasugi et al. [Y]. Their definition is defined by using uniform neighbourhood system. We call the computability defined by their definition the computability in Yasugi's sense, or Y-computability.

They discuss open neighbourhood system in the literature [Y], although the neighbourhood system defined as in Lemma 2.14 is not always an open neighbourhood system. Therefore we sometimes replace $\{V_i(x)\}$ with $\{\mathcal{G}(V_i(x))\}$ in the statements given by them. Moreover, instead that they defined computable functions only for the functions into real numbers, we generalise the range of their computable functions into general effective uniform space,

Definition 6.2 (Computable function in Yasugi's sense) Let $X = (|X|, \{V_i^X(x)\}_{i,x}, S_X)$ and $Y = (|Y|, \{V_j^Y(y)\}_{j,y}, S_Y)$ be triples which consist of underlying space X and Y, effective uniform neighbourhood systems $\{V_i^X(x)\}$ and $\{V_j^Y(y)\}$, and computability structures S_X and S_Y.

A function $f : X \to Y$ is *computable in Yasugi's sense*, or *Y-computable*, iff the followings hold:

1. For $s \in S_X$, $f \circ s \in S_Y$.
2. For any $s \in S_X$, there is a total recursive function $j_s : \mathbb{N} \to \mathbb{N}^\omega$ such that $f(V^X_{j_s(i,n)}(x_i)) \subset V_n^Y(f(x_i))$ for any i and n.

3. There are $a \in S_X$ and a total recursive function $j_a : \mathbb{N} \to \mathbb{N}^\omega$ such that

(3.1) The range $a(\mathbb{N})$ is dense in X.

(3.2) $f(V^X_{j_a(i,n)}(x_i)) \subset V^Y_n(f(x_i))$ for any i and n.

(3.3) $\bigcup_{i\in\mathbb{N}} \mathcal{G}(V^X_{j_a(i,n)}(x_i)) = X$.

Lemma 6.3 *Let X be an algebraic structure with effective uniform topology, and LM be the effective limit of it. Put $\{V_i(x)\}$ be the effective uniform neighbourhood system defined as Lemma 2.14. Let S_X be a computable structure over X. If a function of $X \to Y$ is computable, then it is Y-computable.*

Lemma 6.4 *Let X be an algebraic structure with effective uniform topology, and LM be the effective limit of it. Put $\{V_i(x)\}$ be the effective uniform neighbourhood system defined as Lemma 2.14. Let S_X be a computable structure over X. Suppose the following set is recursively enumerable for each $s \in S_X$:*

$\{(i,j,n,n') \in \mathbb{N}^4 \mid \mathcal{F}(V_j(s(n'))) \subset \mathcal{G}(V_i(s(n)))\}$,

$\{(i,j,n,n') \in \mathbb{N}^4 \mid \mathcal{F}(V_j(s(n'))) \not\subset \mathcal{G}(V_i(s(n)))\}$.

If a function of $X \to Y$ is Y-computable, then it is computable.

Outline of the proof. We can obtain the modulus $m : \mathbb{N}^2 \to \mathbb{N}$ of such function effectively under this assumption.

Theorem 6.5 *Under the same assumption as the previous lemma (Lem. 6.4), a function $f : X \to Y$ is computable iff it is Y-computable.*

Proof. It is directly derived form the previous two lemmata (Lem. 6.3 & 6.4). ∎

Remark 6.6 The algebraic structure $\mathbb{R}$ satisfies the assumption of the previous lemma (Lem. 6.4). Therefore, for each function of $\mathbb{R} \to Y$, it is computable iff it is Y-computable.

Remark 6.7 This condition appears in the definition of computability by Yasugi et al. [Y]:

$$\bigcup_{i\in\mathbb{N}} \mathcal{G}(V^X_{j_a(i,n)}(x_i)) = X.$$

The function j_a corresponds to the modulus m in our definition.

In their definition, the condition asserts only the existence of i such that $x \in V^X_{j_a(i,n)}(x_i)$ for x, but does not asset any effective way to obtain such i. Our definition asserts the effectivity of such i, because our definition asserts the numerical inequality, which is recursively justified.

7 Conclusional Remark

The essential definition of this work is Definition 5.8, which defines computable functions over algebraic structures with effective uniform topology. The main theorem (Thm. 5.17) says that the computability in our definition is equivalent to that in the definition by Weihrauch [W] with some suitable assumptions. And also, Theorem 6.5 says that our computability is equivalent to that by Yasugi and others, with some suitable assumptions.

Definition 5.8 has some conditions on the domain of the function. The author think that such conditions on the domain seem inessential. The author would like to make this condition more natural as a future work.

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