Effective Fixed Point Theorem Over a Non-computably Separable Metric Space

Takeuti Izumi

Graduate School of Informatics, Kyoto University
606-8501 JAPAN
takeuti@kuis.kyoto-u.ac.jp

Abstract. This paper shows effective fixed point theorems for computable contractions. Effective fixed point theorem for computable contractions over a computable metric space is easily shown. A function over a computable metric space is represented by a Type-1 function, and the fixed point of a contraction is given by iteration of such Type-1 function. If the contraction is computable, then its fixed point is also computable. If the support space is not computably separable, the method above is not available. The function space of an interval into real numbers is not computably separable with polynominal time computability. This paper show the fixed point theorem for such non-computably separable spaces. This theorem is proved with iteration of Type-2 functionals. As an example of that, this paper shows that Takagi function is a polynominal time computable function.

1 Introduction

When we discuss computable objects in a metric space, we often define the computability and computational complexity of such objects with a separating function, as is studied by Weihrauch [Weihrauch93, Weihrauch00] and others. Our aim is to show two theorems on fixed points of computable contractions. One of the theorems deals with a metric space with a separating function, and the other deals with a metric space without a separating function.

The computability means the computability by total recursive functions in the works of Weihrauch [Weihrauch93, Weihrauch00] or others, for example, by Brattka [Brattka]. We would like to discuss polynominal time computability in this paper.

We use the representations in discussing computational complexity of mathematical objects. A representation is an elements of some structure over natural numbers, for example, $\mathbf{N} \to \mathbf{N}$, $(\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N})$, and so forth. Therefore, we can define computational complexity for representations naturally. We will use the following notation for the relation of mathematical objects and their representations:

$$\text{representation} \vdash \text{represented object}.$$

We will define some representing systems, which are for points of metric spaces, functions over metric spaces, and functionals of functions into functions. We expect a representing system to satisfy the following two properties:

(Existence of the representation) For each x, there is some r such that $r \vdash x$.
(Uniqueness of the represented object) For each x, x' and r, if $r \vdash x$ and $r \vdash x'$ then $x = x'$.
An object has many representations of it in general. We have not yet proved the existence of representation for functionals, although we have the conjecture that this property holds under some adequate assumption.

In Section 3, we will show the effective fixed point theorem over computably separable metric spaces, which is Theorem 1. This theorem states that a computable contraction over a computably separable metric space has a computable fixed point.

We write $I \stackrel{conti}{\longrightarrow} \mathbf{R}$ for the set of all the continuous functions of the unit interval $I = [0, 1]$ into $\mathbf{R}$, and we write $\mathcal{C}_{I,\mathbf{R}}$ for the set of all the computable functions in $I \stackrel{conti}{\longrightarrow} \mathbf{R}$ under a computational complexity $\mathcal{C}$. The function space $\mathcal{C}_{I,\mathbf{R}}$ is computably separable if the complexity class $\mathcal{C}$ is stronger than or equal to E^3, which is the third class of recursive functions by Grzegorczyk [Grzegorczyk]. As is shown in Proposition 4.8, if the complexity class $\mathcal{C}$ is equal to polynominal time computability, we do not know the adequate separating function for the computable functions in $\mathcal{C}_{I,\mathbf{R}}$. Nevertheless, we discuss the polynominal time computability of objects over the function space $I \stackrel{conti}{\longrightarrow} \mathbf{R}$.

In Section 4, We will give a sufficient condition for the claim that the fixed point of contraction over the function space $I \stackrel{conti}{\longrightarrow} \mathbf{R}$ is computable, which is Theorem 2. We will show a contraction whose fixed point is Takagi function, which is explained in the literature [YHK], and prove that the contraction satisfies the fixed point theorem. Thus, we show that Takagi function is polynominal time computable.

2 Foundation

Notation 2.1 Let $\mathbf{N} = \{0, 1, 2, ...\}$ be the set of 0 and positive integers. The elements of $\mathbf{N}$ are called *natural numbers*.

Notation 2.2 We write P for the set of all the polynominal time computable functions over $\mathbf{N}$. A function f belongs to P iff there is a Turing machine which computes the binary expression of $f(x)$ from the binary expression of x in time $O((\log x)^k)$ for some k independent to x. The input size is as large as $\log x$.

Notation 2.3 We write E^3 for the third class of primitive recursive functions by Grzegorczyk [Grzegorczyk]. A function f belongs to E^3 iff it is computable in time $O(\exp^k n)$ for the input size n and some constant k independent to n. The notation $\exp^k$ is defined as $\exp^0 n = n$ and $\exp^{k+1} n = \exp^k(\exp n)$.

Notation 2.4 We write PR for the set of all the primitive recursive functions.

Notation 2.5 The classes P, E^3 and PR are called *computational complexity classes*. We use $\mathcal{C}$ as a variable which ranges over P, E^3 and PR. The notation $\mathcal{C}_{\mathbf{N}^m,\mathbf{N}^n}$ stands for the set of all functions of $\mathbf{N}^m$ into $\mathbf{N}^n$ in the class $\mathcal{C}$.

Definition 2.6 ($\mathcal{C}$-function) A functions in $\mathcal{C}_{\mathbf{N}^m,\mathbf{N}^n}$ is called a *$\mathcal{C}$-function.*

Remark 2.7 Here we will give the standard encoding of pairs and lists of natural numbers. We will use them in encoding connective segments in Definition 4.2. There are many ways to encode pairs, or lists, and there are no essential differences among them. The essential properties are Remarks 2.9 for pairs, and Remark 2.11 for lists.

Notation 2.8 The functions $left, right : \mathbf{N} \to \mathbf{N}$ and $pair : \mathbf{N}^2 \to \mathbf{N}$ is defined as follows. Here m and n are natural numbers.

$left(0) = 0$, $left(2n) = 2right(n)$, $left(2n+1) = 2right(n)+1$,
$right(2n) = right(2n+1) = left(n)$,
$pair(0,0) = 0$,
$pair(2m,n) = 2pair(n,m)$, $pair(2m+1,n) = 2pair(n,m)+1$.

Remark 2.9 The functions $left, right$ and $pair$ are P-functions, which play the roles of encoding and decoding of pairing. Indeed, we have the following equations:

$$n = pair(left(n), right(n)), m = left(pair(m,n)), n = right(pair(m,n)).$$

Notation 2.10 We write $\langle n_1, n_2, ..., n_l \rangle$ for the code of the list $(n_1, n_2, ..., n_l)$, which is defined as follows:

$$\underbrace{\langle\,\rangle}_{\text{empty list}} = 0, \quad \underbrace{\langle 0,0,...,0\rangle}_{\text{length } l} = 2^{l-1} \ (l \geq 1),$$
$$\langle n_1, n_2, ..., n_{i-1}, 2^m + n_i, n_{i+1}, ..., n_l\rangle = 2^{l(m+1)+i-1} + \langle n_1, n_2, ..., n_i, ..., n_l\rangle \text{ where } n_i < 2^m.$$

Remark 2.11 The following functions belong to P.

(length) $\langle n_1, n_2, ..., n_l\rangle \mapsto l : \mathbf{N} \to \mathbf{N}$
(component) $(i, \langle n_1, n_2, ..., n_l\rangle) \mapsto n_i$ (if $i \leq l$) $: \mathbf{N}^2 \to \mathbf{N}$
(concatenation)
$(\langle n_1, n_2, ..., n_l\rangle, \langle n_{l+1}, n_{l+2}, ..., n_{l'}\rangle) \mapsto \langle n_1, n_2, ..., n_{l'}\rangle : \mathbf{N}^2 \to \mathbf{N}$

Moreover, it holds that if $l \geq 1$ then $2^{l-1} \leq \langle n_1, n_2, ..., n_l\rangle < (4 \max_i n_i)^l$.

3 Metric Space

Definition 3.1 (Numbered metric space) A triple $X = (|X|, d, s)$ is an *numbered metric space* iff $(|X|, d)$ is a metric space, and s is a function of $\mathbf{N}$ into $|X|$ such that the image $s(\mathbf{N})$ is dense in $|X|$. We sometimes write X for $|X|$. The function s is called a *separating function.*

Definition 3.2 (Computable metric space) A numbered metric space $X = (|X|, d, s)$ is a *computable metric space* under computational complexity $\mathcal{C}$ iff there is a $\mathcal{C}$-function $f : \mathbf{N}^3 \to \mathbf{N}$ which satisfies the following condition:

$$\frac{f(m,n,n') - 1}{m} \leq d(s(n), s(n')) \leq \frac{f(m,n,n') + 1}{m} \text{ if } m \geq 1.$$

A computable metric space under computational complexity $\mathcal{C}$ is also called a *$\mathcal{C}$-metric space.*

Remark 3.3 There have been several definitions of computable metric spaces. The definition used by Brattka [Brattka] is equivalent to this definition where the computational complexity is Turing computability. This definition above is stronger than the definition which was written by Weihrauch [Weihrauch93].

Example 3.4 Let $|\cdot - \cdot|$ be the function of $\mathbf{R}^2$ into $\mathbf{R}$ which maps (x, y) into $|x - y|$. Let $s_{\mathbf{R}}$ be the function of $\mathbf{N}$ into $\mathbf{R}$ such that

$s_{\mathbf{R}}(0) = 0$

$s_{\mathbf{R}}(4^n + m) = (m - 2^{2n+1})/2^n$ where $0 \leq m < 3 \cdot 4^n$

Then, $(\mathbf{R}, |\cdot - \cdot|, s_{\mathbf{R}})$ is a P-metric space.

Example 3.5 Let I be the unit interval $[0, 1]$. Let $|\cdot - \cdot|$ be the distance function above. Let s_I be the function of $\mathbf{N}$ into I such that

$s_I(0) = 0$

$s_I(2^n + m) = (2m + 1)/2^{n+1}$ where $0 \leq m < 2^n$

Then, $(I, |\cdot - \cdot|, s_I)$ is a P-metric space.

Notation 3.6 Hereafter we write $\mathbf{R}$ and I for the triples above, and regard them as numbered metric spaces.

Definition 3.7 (Representation of a point) Let $X = (X, d, s)$ be a numbered metric space. For a function $f : \mathbf{N} \to \mathbf{N}$ and a point $x \in X$, the function f *represents* the point x iff for each $n \in \mathbf{N}$ if $n \geq 1$ then $d(x, s(f(n))) \leq 1/n$. We write $f \vdash x$ when f represents x.

Lemma 3.8 (Existence) *For each $x \in X$, there is a function $f : \mathbf{N} \to \mathbf{N}$ such that $f \vdash x$.*

Proof. Because $s(\mathbf{N})$ is dense in X. □

Lemma 3.9 (Uniqueness) *For each $x, x' \in X$, if $f \vdash x$ and $f \vdash x'$ for the same function $f : \mathbf{N} \to \mathbf{N}$, then $x = x'$.*

Proof. Obvious. □

Definition 3.10 ($\mathcal{C}$-point) Let $X = (|X|, d, s)$ be a $\mathcal{C}$-metric space. A point $x \in X$ is a *$\mathcal{C}$-point* iff there is a $\mathcal{C}$-function $f : \mathbf{N} \to \mathbf{N}$ such that $f \vdash x$. We write $\mathcal{C}(X)$ for the set of all the $\mathcal{C}$-points of X.

Definition 3.11 ($\mathcal{C}$-fundamental sequence) Let $X = (|X|, d, s)$ be a $\mathcal{C}$-metric space. A sequence of points $\{x_1, x_2, ...\}$ ($x_n \in X$ for each n) is a *$\mathcal{C}$-fundamental sequence* in X iff $d(x_m, x_n) \leq 1/m + 1/n$ for each $m, n \in \mathbf{N}$ and that there is a function $f \in \mathcal{C}$ such that $x_n = s(f(n))$ for each n.

Remark 3.12 A point $x \in X$ is a $\mathcal{C}$-point iff there is a $\mathcal{C}$-fundamental sequence $\{x_n\}_n$ such that $|x - x_n| \leq 1/n$ for each n.

Definition 3.13 ($\mathcal{C}$-complete) A $\mathcal{C}$-metric space X is *$\mathcal{C}$-complete* iff each $\mathcal{C}$-fundamental sequence in X converges.

Definition 3.14 ($\mathcal{C}$-dense) The separating function s is *$\mathcal{C}$-dense* iff all the points in X are $\mathcal{C}$-points.

Definition 3.15 (Computably separable) A $\mathcal{C}$-metric space is computably separable under computational complexity $\mathcal{C}$, or *$\mathcal{C}$-separable*, iff the separating function is $\mathcal{C}$-dense. A $\mathcal{C}$-separable $\mathcal{C}$-metric space is called *$\mathcal{C}$-separable metric space*.

Remark 3.16 **(Polynominal time computable point)** The definition of P-points above is quite natural. If $f \vdash x \in \mathbf{R}$ then $s(f(2^l))$ is an approximation of x with error $O(2^{-l})$. Thus it gives the significant digits of size $O(l)$. This fact means that we can calculate an output of size l in polynominal time of l. This is a nice symmetry for P-function, that is, if f is a P-function then we can calculate the value for an input of size l in polynominal time of l.

Remark 3.17 **(Harmonic modulus)** The conditions of convergence above are written as $d(x, s(f(n))) \leq 1/n$ or $d(x, x_n) \leq 1/n$. We would like to call this modulus $(-)_n \leq 1/n$ *harmonic modulus*. On the other hand, the condition of convergence is written as $d(x, x_n) \leq 2^{-n}$ in many literatures. We would like to call this modulus $(-)_n \leq 2^{-n}$ *geometric modulus*.

If we use the geometric modulus in describing the definition of P-points, we say that a point x is P-point iff there is a function $f : \mathbf{N} \to \mathbf{N}$ such that $d(x, s(f(n))) \leq 2^{-n}$ and $f(n)$ can be calculate in time $O(n^k)$ for some k independent to n.

It seems more natural to use harmonic modulus than geometric modulus in defining P-points. That is because it seems more easy to say that f is P-function than that $f(n)$ can be calculate in time $O(n^k)$. Moreover, 2^n is not P-function anyway.

Remark 3.18 For the notions defined above except for completeness, a notion for P implies that for E^3, and that for E^3 implies that for PR. Namely, a P-function is an E^3-function, and an E^3-function is a PR-function. The inverses do not hold generally. The opposite direction holds for the notion of completeness. A PR-complete space is E^3-complete, and an E^3-complete space is P-complete.

Remark 3.19 Let $X = (X, d, s)$ be a $\mathcal{C}$-metric space. Let Y be a subset such that $s(\mathbf{N}) \subset Y \subset X$. Then (Y, d, s) is also a $\mathcal{C}$-metric space. The $\mathcal{C}$-metric space (Y, d, s) is $\mathcal{C}$-complete iff $Y \supset \mathcal{C}(X)$. The $\mathcal{C}$-metric space (Y, d, s) is $\mathcal{C}$-separable iff $Y \subset \mathcal{C}(X)$.

Definition 3.20 (Representation of a function) Let $X = (|X|, d_X, s_X)$ and $Y = (|Y|, d_Y, s_Y)$ be numbered metric spaces. Let f, g and h be functions such as $f : |X| \to |Y|$ and $g, h : \mathbf{N}^2 \to \mathbf{N}$. Then, the pair of functions (g, h) *represents* the function f iff the following conditions hold:

$$\forall m, n.\ m \neq 0 \Rightarrow d_Y(f(s_X(n)), s_Y(g(m, n))) \leq 1/m$$

$$\forall m, n \in \mathbf{N}. \forall x, y \in |X|,$$
$$m \neq 0,\ d_X(x, s_X(0)) \leq n,\ d_X(y, s_X(0)) \leq n,\ d_X(x, y) \leq h(m, n)$$
$$\Rightarrow d_Y(f(x), f(y)) \leq 1/m$$

If a pair of functions (g,h) represents a function f then we write $(g,h) \vdash f$.

For $h' : \mathbf{N} \to \mathbf{N}$, $(g,h') \vdash f$ iff $(g,h) \vdash f$, where the function h is put as $h(m,n) = h'(m)$. We also say that (g,h') *represents* f.

Remark 3.21 The first component of the representation mentions the approximation of the value over the separating set, and the second component mentions the modulus of locally uniform continuity.

Remark 3.22 Let f be a function of X to Y. If X is bounded, then the following two conditions are equivalent.

(1) There are $g, h : \mathbf{N}^2 \to \mathbf{N}$ such that $(g,h) \vdash f$.

(2) There are $g : \mathbf{N}^2 \to \mathbf{N}$ and $h' : \mathbf{N} \to \mathbf{N}$ such that $(g,h') \vdash f$.

If X is not bounded, it does not always hold that the two condition above are equivalent.

Hereafter, we discuss functions of bounded domains mainly. Thus we often deal with the representation (g,h') in the condition (2) above.

Remark 3.23 Weihrauch defined the notion of computable functions with Type-2 machines [Weihrauch00]. As for functions over $\mathbf{R}$, or over compact spaces, his definition and our definition are equivalent to each other, although our definition uses functions $g, h : \mathbf{N}^2 \to \mathbf{N}$, which are called Type-1 object. Our definition implies his definition in general. We will discuss this point later in Section 5.2.

Lemma 3.24 (Existence) *Let X and Y be numbered metric spaces. Suppose that each bounded closed set in X is compact. Then each continuous function $f : X \to Y$ has a representation g,h such that $g, h : \mathbf{N}^2 \to \mathbf{N}$ and $(g,h) \vdash f$.*

If X is compact, then each continuous function $f : X \to Y$ has a representation (g,h) such that $g : \mathbf{N}^2 \to \mathbf{N}$, $h : \mathbf{N} \to \mathbf{N}$ and $(g,h) \vdash f$.

Proof. The function g exists because there is a fundamental sequence $s_Y(y_n)$ which converges into $s_Y(f(x))$ for each $x \in X$. The function h exists because a continuous function of a compact domain is uniformly continuous. □

Lemma 3.25 (Uniqueness) *Let X and Y be computable metric spaces. If two continuous functions $f, f' : X \to Y$ have the same representation, that is, $(g,h) \vdash f$ and $(g,h) \vdash f'$, then $f = f'$.*

Proof. That is because f and f' are continuous, $f(s_X(n)) = f'(s_X(n))$ for each n, and $s_X(\mathbf{N})$ is dense in X. □

Definition 3.26 (Computable function) Let $X = (|X|, d_X, s_X)$ and $Y = (|Y|, d_Y, s_Y)$ be computable metric spaces. A function $f : |X| \to |Y|$ is a *$\mathcal{C}$-function* of X into Y iff there is functions $g, h \in \mathcal{C}_{\mathbf{N}^2,\mathbf{N}}$ such that $(g,h) \vdash f$. We write $\mathcal{C}_{X,Y}$ for the set of all $\mathcal{C}$-functions of X into Y.

Proposition 3.27 *For $x \in \mathcal{C}(X)$ and $f \in \mathcal{C}_{X,Y}$, the value $f(x)$ belongs to $\mathcal{C}(Y)$. For $f \in \mathcal{C}_{X,Y}$ and $g \in \mathcal{C}_{Y,Z}$, the composition $g \circ f$ belongs to $\mathcal{C}_{X,Z}$.*

Proof. Easy. □

Definition 3.28 (Contraction) For a metric space (X, d), a function $f : X \to X$ is a *contraction of maximum magnification* α iff $\alpha < 1$ and for any $x, y \in X$, $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$.

Theorem 1. (Fixed point theorem over numbered metric spaces) *Let $X = (X, d, s)$ be a $\mathcal{C}$-complete $\mathcal{C}$-metric space, $f \in \mathcal{C}_{X,X}$ be a contraction, and $\bar{f} : \mathbf{N}^2 \to \mathbf{N}$ be a $\mathcal{C}$-function such that*

$$\forall m, n.\ m \neq 0 \Rightarrow d(f(s_X(n)), s(\bar{f}(m, n))) \leq 1/m$$

Suppose that there is a sequence of functions $g_k \in \mathcal{C}$ such that if $d(s(n), s(0)) \leq k$ then $\bar{f}(m, n) \leq g_k(m)$.

Then there is a $\mathcal{C}$-point x such that $x = f(x)$.

Proof. The proof consists of several steps.

Step 1. The function $h : \mathbf{N} \to \mathbf{N}$ is defined by the induction below:

$$h(0) = h(1) = 0$$
$$h(2n) = h(2n + 1) = \bar{f}(2^i, h(n)) \quad (i \in \mathbf{N},\ 2^i \leq n < 2^{i+1})$$

It holds that $h(2^n) = h(2^n + m)$ if $0 \leq m < 2^n$.

Step 2. Put $x_n = s(h(2^n))$ and $L = d(x_0, f(x_0))$. Then the followings hold.

$$d(x_n, f^n(x_0)) \leq 1/2^{n-1} + \alpha/2^{n-2} + \cdots + \alpha^{n-2}/2 + \alpha^{n-1}$$

$$\begin{cases} = \dfrac{\alpha^n - 2^{-n}}{\alpha - 1/2} & (\alpha \neq 1/2) \\ = n2^{1-n} & (\alpha = 1/2) \end{cases}$$

$$d(f^n(x_0), f^{n'}(x_0)) \leq \alpha^n L + \alpha^{n+1} L + \cdots + \alpha^{n'-2} L + \alpha^{n'-1} L$$
$$= (\alpha^n - \alpha^{n'}) L/(1 - \alpha) \quad (n \leq n')$$

Step 3. Put k, k' be integers such that

$$k \geq 2,\ \alpha^k \geq 1/2$$

and that

$$2^{k'} \geq (L/(1 - \alpha)) + (1/(\alpha - (1/2)))$$

or

$$2^{k'} \geq 2L + 2 \ \text{ if } \alpha = 1/2.$$

Then it holds that

$$d(x_n, x_{n'}) \leq 2^{-(n/k)+k'} + 2^{-(n'/k)+k'}.$$

Step 4. For n, an integer i_n is put as $2^{i_n} \leq 2^{k'+1} n^k < 2^{i_n+1}$. Then $h(2^{i_n}) = h(2^{k'+1} n^k)$ and $n < 2^{(i_n/k)-k'}$. Therefore, $d(x_{i_n}, x_{i_{n'}}) < 1/n + 1/n'$. Thus, $\{x_{i_n}\}_n$ is a fundamental sequence.

Step 5. For each $n \in \mathbf{N}$,

$$d(x_0, x_n) \leq d(x_0, f^n(x_0)) + d(x_n, f^n(x_0)) \leq 1 + \alpha L/1 - \alpha.$$

Hence for each $n \in \mathbf{N}$,

$$d(s(0), s(h(n))) \leq 1 + \alpha L/1 - \alpha.$$

Put $l \geq 1 + \alpha L/1 - \alpha$. Then, the value of h is upper-bounded by a $\mathcal{C}$-function g_l. Hence the recursion above is a limited recursion, that is, the function h is a $\mathcal{C}$-function.

Step 6. The sequence $\{x_{i_n}\}_n$ is a $\mathcal{C}$-fundamental sequence. Because X is $\mathcal{C}$-complete, this sequence $\{x_{i_n}\}_n$ converges at a point x. The sequence $\{f^n(x_0)\}_n$ also converges at x. Therefore x is a fixed point of f. □

Remark 3.29 For each $\mathcal{C}$-function over I or over $\mathbf{R}$, there always exists the functions $g_k \in \mathcal{C}$ in the statement above. Therefore, the fixed point theorem always holds in I and in $\mathbf{R}$.

Lemma 3.30 (Fixed point theorem in PR**)** *Let $X = (X, d, s)$ be a* PR*-complete* PR*-metric space, and $f \in \mathrm{PR}_{X,X}$ is a contraction. Then, there is a* PR*-point x such that $x = f(x)$.*

Proof. In the proof of previous lemma, the recursion always defines a PR-function without the evidence of the upper-bounding function. □

4 Function Space

Notation 4.1 We write $I \stackrel{conti}{\longrightarrow} \mathbf{R}$ for the set of continuous functions of I into $\mathbf{R}$, and regard $I \stackrel{conti}{\longrightarrow} \mathbf{R}$ as a metric space, with a distance $d : f, g \mapsto d(f, g) = \max_{x \in I} |g(x) - f(x)|$, which is the distance induced from sup-norm.

Definition 4.2 (Connected segment) Let $q = ((x_0, y_0), (x_1, y_1), ..., (x_n, y_n))$ be a sequence of $n + 1$ points in $I \times \mathbf{R}$. Suppose that it satisfies the followings:

$$x_0 = 0,\ x_n = 1,\ x_i < x_{i+1} \text{ for } 0 \leq i \leq n - 1$$

Then this q determines the function $f : I \stackrel{conti}{\longrightarrow} \mathbf{R}$ as below:

$$f(x) = y_i + (y_{i+1} - y_i)\frac{x - x_i}{x_{i+1} - x_i} \quad (x_i \leq x \leq x_{i+1})$$

We call functions of this form *connected segments.*

Let s_I and $s_{\mathbf{R}}$ be separating functions of I and $\mathbf{R}$. Then we will define a separating function s for $I \stackrel{conti}{\longrightarrow} \mathbf{R}$ by using connected segments. For $n \in \mathbf{N}$, there is a unique sequence $(n_1, n_2, ..., n_l) \in \mathbf{N}^*$ such that $N = \langle n_1, n_2, ..., n_l \rangle$. For each component n_i, we assign a point $(x_i, y_i) = (s_I(\mathit{left}(n_i)), s_{\mathbf{R}}(\mathit{right}(n_i))) \in I \times \mathbf{R}$. If this sequence $((x_1, y_1), (x_2, y_2), ..., (x_l, y_l))$ satisfies the condition:

$$x_1 = 0,\ x_l = 1,\ x_i < x_{i+1} \text{ for } 1 \leq i \leq l - 1$$

Then $s(n)$ is a function which is determined by this sequence. Otherwise, $s(n)$ is a constant function such as $s(n)(x) = 0$.

Notation 4.3 Hereafter we fix the notation d for the distance of $I \stackrel{conti}{\longrightarrow} \mathbf{R}$ which is induced from sup-norm, and s for this enumeration of connected segments as the separating function of $I \stackrel{conti}{\longrightarrow} \mathbf{R}$.

Remark 4.4 There are P-functions $\bar{s} : \mathbf{N}^3 \to \mathbf{N}$ and $\tilde{s} : \mathbf{N}^2 \to \mathbf{N}$ such that

$$\forall i, m, n.\ m \neq 0 \Rightarrow |s(i)(s_I(n)) - s_{\mathbf{R}}(\bar{s}(i, n, m))| \leq 1/m$$

$$\forall i, m \in \mathbf{N}. \forall x, y \in I.\ m \neq 0,\ |y - x| \leq \tilde{s}(i, m) \Rightarrow |s(i)(y) - s(i)(x)| \leq 1/m$$

Thus, $s(i) \in \mathrm{P}_{I,\mathbf{R}}$.

Remark 4.5 There are functions $\bar{d} : \mathrm{P}_{\mathbf{N}^3,\mathbf{N}}$ such that
$$\bar{d}(m,n,n') - 1 \leq m \cdot d(s(n), s(n')) \leq \bar{d}(m,n,n') + 1$$
Thus, $(I \overset{conti}{\longrightarrow} \mathbf{R}, d, s)$ is a P-metric space.

Notation 4.6 We write $I \overset{conti}{\longrightarrow} \mathbf{R}$ for a numbered metric space $(I \overset{conti}{\longrightarrow} \mathbf{R}, d, s)$.

Proposition 4.7 *If $\mathcal{C} = \mathrm{E}^3$ or $\mathcal{C} = \mathrm{PR}$, then $\mathcal{C}_{I,\mathbf{R}} = \mathcal{C}(I \overset{conti}{\longrightarrow} \mathbf{R})$. Thus, the $\mathcal{C}$-metric space $(\mathcal{C}_{I,\mathbf{R}}, d, s)$ is $\mathcal{C}$-complete and $\mathcal{C}$-separable.*

Proof. Easy. □

Proposition 4.8 $\mathrm{P}(I \overset{conti}{\longrightarrow} \mathbf{R}) \subset \mathrm{P}_{I,\mathbf{R}}$ *and* $\mathrm{P}(I \overset{conti}{\longrightarrow} \mathbf{R}) \neq \mathrm{P}_{I,\mathbf{R}}$. *Thus, the* P*-metric space* $\mathrm{P}_{I,\mathbf{R}}$ *is* P*-complete but not* P*-separable.*

Proof. In order to make a connected segments as an approximation of a curve with error $\leq 1/n$, we have to make the connected segment of size $O(\sqrt{n})$ in general. On the other hand, a P-fundamental sequence has a connected segment of size $O((\log n)^k)$ with error $\leq 1/n$. The size $O(\sqrt{n})$ is greater than $O((\log n)^k)$. □

Remark 4.9 The previous lemma states for the special separating function which is defined by the connective segments. We have a further conjecture as below. Indeed, we have not proved this conjecture yet. We cannot regard $\mathrm{P}_{I,\mathbf{R}}$ as P-separable metric space before we find out a separating function which is dense in $\mathrm{P}_{I,\mathbf{R}}$.

Conjecture 4.10 No separating function $s : \mathbf{N} \to (I \overset{conti}{\longrightarrow} \mathbf{R})$ is P-dense in a metric space $(\mathrm{P}_{I,\mathbf{R}}, d)$, where d is the distance defined above.

Notation 4.11 The operation T over $I \overset{conti}{\longrightarrow} \mathbf{R}$ is defined as follows:
$$T(f)(x) = \begin{cases} \dfrac{f(2x)}{2} + x & (0 \leq x \leq 1/2) \\ \dfrac{f(2-2x)}{2} + 1 - x & (1/2 \leq x \leq 1) \end{cases}$$
This operation T is a contraction with magnification $1/2$, and the fixed point of T is called Takagi function.

Proposition 4.12 *There is a* P*-function f such that $T(s(n)) = s(f(n))$, where $s(\,)$ is the enumeration of the connected segments.*

Proof. Easy. □

Lemma 4.13 *Takagi function is* E^3*-computable. Hence, it is* PR*-computable.*

Proof. Because $f \in \mathrm{P}$, the iterating function $(m,n) \mapsto f^m(n)$ belongs to E^3. Then, so does the function $(2^m, n) \mapsto f^m(n)$. Therefore, by the fixed point theorem (Theorem 1), Takagi function is in $\mathrm{E}^3(I \overset{conti}{\longrightarrow} \mathbf{R})$. □

Remark 4.14 It is easy to observe that Takagi function $t(\ )$ is in $\mathrm{P}_{I,\mathbf{R}}$. We will show this fact as the consequence of fixed point theorem. Unfortunately, the operator T does not satisfies the hypothesis of the fixed point theorem (Theorem 1), because there are no P-functions g_k in the theorem. Actually, Takagi function $t(\ )$ is not in $\mathrm{P}(I \stackrel{conti}{\longrightarrow} \mathbf{R})$. Therefore we will show that there is a functional

$$(\bar{T}, \tilde{T}) : (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N}) \longrightarrow (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N})$$

such that $(\bar{T}, \tilde{T})^n(\bar{f}, \tilde{f})$ represents a function t_n, and that t_n converges into $t(\ )$ as $n \to \infty$.

Definition 4.15 (Functional) A function over functions is called a *functional*. For a typical example, $F : (\mathbf{N} \to \mathbf{N}) \to \mathbf{N} \to \mathbf{N}$ is a functional. The computability of functionals is defined by Odifreddi [Odifreddi]. We also regard functions with more general arities as functionals. For example, $F : (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N}) \to (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N})$ is also a functional.

Definition 4.16 (Representation of functionals) Let X_1, X_2, X_3 and X_4 be numbered metric spaces. Suppose that X_1 and X_3 are bounded. Let F be a function in $(X_1 \stackrel{conti}{\longrightarrow} X_2) \to (X_3 \stackrel{conti}{\longrightarrow} X_4)$. Let $\hat{F}$ be a functional in $(\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N}) \to (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N})$. Then, the functional $\hat{F}$ *represents* F iff for any $\bar{f} : \mathbf{N}^2 \to \mathbf{N}, \tilde{f} : \mathbf{N} \to \mathbf{N}$ and $f : X_1 \stackrel{conti}{\longrightarrow} X_2$, if $(\bar{f}, \tilde{f}) \vdash f$ then $\hat{F}(\bar{f}, \tilde{f}) \vdash F(f)$. We write $\hat{F} \vdash F$ when $\hat{F}$ represents F.

Remark 4.17 If the domain X_1 is not bounded, then the representation (g, h) of a function $f : X_1 \to X_2$ has the arity such as $(g, h) : (\mathbf{N}^2 \to \mathbf{N})^2$ in general. This holds also for X_3. Therefore, if neither X_1 nor X_3 is bounded, we have to consider the representation $\hat{F}$ of arity $(\mathbf{N}^2 \to \mathbf{N})^2 \to (\mathbf{N}^2 \to \mathbf{N})^2$. In this paper we discuss only the case that X_1 and X_3 are bounded. Therefore we deal with only the representations of arity $(\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N}) \to (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N})$.

Conjecture 4.18 (Existence) Let X_1, X_2, X_3 and X_4 be numbered metric spaces. Suppose that X_1 and X_3 are compact. Regard $X_1 \stackrel{conti}{\longrightarrow} X_2$ and $X_3 \stackrel{conti}{\longrightarrow} X_4$ as metric spaces with the following distance functions:

$$\text{For } f, g \in X_1 \stackrel{conti}{\longrightarrow} X_2,\ d(f,g) = \max_{x \in X_1} d_{X_2}(f(x), g(x))$$

$$\text{For } f, g \in X_3 \stackrel{conti}{\longrightarrow} X_4,\ d(f,g) = \max_{x \in X_3} d_{X_4}(f(x), g(x))$$

Then, each uniformly continuous function in $(X_1 \stackrel{conti}{\longrightarrow} X_2) \to (X_3 \stackrel{conti}{\longrightarrow} X_4)$ has the representation $\hat{F} = (\bar{F}, \tilde{F})$, and both functionals $\bar{F} : (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N}) \to \mathbf{N}^2 \to \mathbf{N}$ and $\tilde{F} : (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N}) \to \mathbf{N} \to \mathbf{N}$ are continuous.

Lemma 4.19 (Uniqueness) *Let X_1, X_2, X_3 and X_4 be numbered metric spaces. Suppose that X_1 is compact. If $\hat{F} \vdash F$ and $\hat{F} \vdash F'$, then $F = F'$*

Proof. Each f in the domain of F and F' has the representation $(\bar{f}, \tilde{f})$ such that $(\bar{f}, \tilde{f}) \vdash f$, by Lemma 3.24. Thus $\hat{F}(\bar{f}, \tilde{f}) \vdash F(f)$ and $\hat{F}(\bar{f}, \tilde{f}) \vdash F'(f)$. Therefore $F(f) = F'(f)$ because of Lemma 3.25. □

Theorem 2. (Fixed point theorem for functionals) *Let $X = (|X|, d_X, s_X)$ and $Y = (|Y|, d_Y, s_Y)$ be computable metric spaces. Suppose that X is compact and Y is $\mathcal{C}$-complete. We regard $X \overset{conti}{\longrightarrow} Y$ as a metric space with the distance function $f, g \mapsto \max_{x \in |X|} d_Y(f(x), g(x))$. Let F be contraction of $X \overset{conti}{\longrightarrow} Y$ into $X \overset{conti}{\longrightarrow} Y$. Let $\hat{F}$ be a functional such that $\hat{F} \vdash F$.*

Suppose that the following functions f' and f'' are $\mathcal{C}$-functions for some $\bar{g} \in \mathcal{C}_{\mathbf{N}^2,\mathbf{N}}$, $\tilde{g} \in \mathcal{C}_{\mathbf{N},\mathbf{N}}$ and $g : X \overset{conti}{\longrightarrow} Y$ such that $(\bar{g}, \tilde{g}) \vdash g$.

$$f'(2^k, m, n) = \bar{f}_k(m, n),\ f''(2^k, m) = \tilde{f}_k(m) \text{ where } (\bar{f}_k, \tilde{f}_k) = \hat{F}^k(\bar{g}, \tilde{g})$$

Then, there is a function $f \in \mathcal{C}_{X,Y}$ such that f is a fixed point of F.

Proof. It is obvious that the fixed point of F is represented by the functions $(\bar{f}, \tilde{f})$ which are defined as:

$$\bar{f}(m, n) = f'(2^{k+1}, 2m, n) \qquad \tilde{f}(m) = f'(2^{k+1}, 2m)$$

where $2^k < m \leq 2^{k+1}$. □

Notation 4.20 We define functions in class P as follows:

$$h : \mathbf{N}^2 \to \mathbf{N}; \quad |2s_I(n) - s_I(h(m,n))| < 1/m \ (m \geq 1, s_I(n) \leq 1/2)$$
$$|2 - 2s_I(n) - s_I(h(m,n))| < 1/m \ (m \geq 1, s_I(n) \geq 1/2)$$
$$g : \mathbf{N}^3 \to \mathbf{N}; \quad \left| \frac{s_{\mathbf{R}}(n) + s_I(n')}{2} - s_{\mathbf{R}}(g(m,n,n')) \right| < 1/m \ (m \geq 1)$$

and that $g(m, n, n') = O(m^2), h(m, n) = O(m^2)$. This estimation of growth rate is possible, for the definition of s_I and $s_{\mathbf{R}}$.

Notation 4.21 we define functionals $\bar{T}$, $\tilde{T}$, and $(\bar{T}, \tilde{T})$ as follows:

$$\bar{T} : (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N}) \to \mathbf{N}^2 \to \mathbf{N};$$
$$\bar{T}(\bar{f}, \tilde{f})(m, n) = g(4m, \bar{f}(4m, h(\tilde{f}(4m), n)), h(4m, n))$$
$$\tilde{T} : (\mathbf{N} \to \mathbf{N}) \to \mathbf{N} \to \mathbf{N};$$
$$\tilde{T}(\tilde{f})(m) = \max(\tilde{f}(\lceil 2m/3 \rceil), 4m)$$

where $\lceil - \rceil$ is the ceiling function such that $\lceil x \rceil$ is a integer and $\lceil x \rceil - 1 < x \leq \lceil x \rceil$.

$$(\bar{T}, \tilde{T}) : (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N}) \to (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N});$$
$$(\bar{T}, \tilde{T})(\bar{f}, \tilde{f}) = \left(\bar{T}(\bar{f}, \tilde{f}), \tilde{T}(\tilde{f})\right)$$

Proposition 4.22 $(\bar{T}, \tilde{T}) \vdash T$

Notation 4.23 We write $\hat{0}$ for the constant functions with the constant value 0 for any arities.

Proposition 4.24 *The following functions t' and t'' belong to $\mathrm{P}_{\mathbf{N}^2,\mathbf{N}}$ and to $\mathrm{P}_{\mathbf{N},\mathbf{N}}$.*

$$t'(2^k, m, n) = \bar{f}_k(m, n),\ t''(2^k, m, n) = \tilde{f}_k(m)$$

where $(\bar{f}_k, \tilde{f}_k) = (\bar{T}, \tilde{T})^k(\hat{0}, \hat{0})$

Proposition 4.25 *Easy.*

Corollary 4.26 *Takagi function is in $\mathrm{P}_{I,\mathbf{R}}$.*

Proof. By Theorem 2, Proposition 4.22 and Proposition 4.24. □

5 Discussion

5.1 Related works

There have been several works on computability of objects in some metric spaces. There are two ways to define the computability over a metric spaces.

One is to use computable structure, which is given by Pour-El and Richard [Pour-El&Richards]. For a metric space $X = (|X|, d)$, a computable structure S is a set of sequences of points in $|X|$ which satisfies some properties. One of the properties is that for each computable function $f : \mathbf{N} \to \mathbf{N}$ and a sequence $\{x_n\}_n \in S$, the new sequence $\{x_{f(n)}\}_n$ also belongs to S. Then we will regard that a objects is computable iff the objects is defined by elements of S. For example, a point $x \in |X|$ is computable iff there are a sequence $\{x_n\}_n \in S$ such that the sequence $\{x_n\}$ converges into x with the computable modulus $f : \mathbf{N} \to \mathbf{N}$, that is, it holds that $d(x, x_n) \leq 1/m$ if $n \geq f(m)$ and $m \geq 1$. When we use a computable structure S for a metric space $X = (|X|, d)$, we have to guarantee in advance that all the elements in S are computable.

Another way to define the computability is to use the separating function, which is given by Weihrauch [Weihrauch93]. A triple $X = (|X|, d, s)$ is a computable metric space iff $(|X|, d)$ is a metric space, and $d : |X|^2 \to \mathbf{R}$ is computable with respect to $s : \mathbf{N} \to |X|$. We can guarantee the computability of a sequence $\{x_n \in |X|\}_n$ by using this separating function s, as far as s is adequate to computability. Thus, the sequence $\{x_n\}_n$ is computable iff there is a computable function $f : \mathbf{N}^2 \to \mathbf{N}$ and $s(f(m, n))$ converges into x_n as $m \to \infty$ with a computable modulus.

Our discussion in Section 4 shows that we can discuss computability without the separating function. Therefore, our work suggests that there are many other ways to discuss the computability of the elements of computable structure.

All the works above discussed computability without the limitation of complexity. Our work intends to discuss polynominal time computability over metric space.

5.2 Type-1 objects and Type-2 objects

Functions over natural numbers are called *Type-1 objects*. A function in $\mathbf{N}^2 \to \mathbf{N}$ or in $\mathbf{N}^3 \to \mathbf{N}$ is also called a Type-1 object. A pairing of Type-1 objects is also called a Type-1 object, such as $(\bar{f}, \tilde{f}) : (\mathbf{N}^2 \to \mathbf{N}) \times (\mathbf{N} \to \mathbf{N})$. Functionals, which is functions over functions, are called Type-2 objects. For example, $F : (\mathbf{N} \to \mathbf{N}) \to \mathbf{N} \to \mathbf{N}$ is a Type-2 object. The machines which represents Type-2 objects are called Type-2 machines, The word of Type-2 machine appears in the works by Weihrauch [Weihrauch00].

A computable point in a computable metric space is represented by a fundamental sequence which converges into the point. Hence, computable points in a computable metric space are represented by Type-1 objects.

The representation of a function f over a metric space plays the role of a function over Type-1 objects, namely, the representation of a point x is mapped

to the representation of $f(x)$. Therefore, it is natural that a Type-2 function represents this function f. Although, computable functions over computable metric spaces are represented by Type-1 object. That is because the domain of the function is separable, so we can restrict the arguments into only the points in the separating set.

On the other hand, we should represent the functions over the function space $\mathrm{P}_{I,\mathbf{R}}$. by functionals, which is Type-2 object. That is because we does not have a separating function for the function space $\mathrm{P}_{I,\mathbf{R}}$.

Acknowledgements

The author would like to thank Yasugi Mariko, Kamo Hiroyasu, and anonymous referees for discussions and comments. This work was presented at the Forth Workshop on Computability and Complexity in Analysis. The author also thank the participants of the workshop for the comments.

References

[Brattka] Brattka, Vasco: Computable invariance. *Theoret. Comput. Sci.* 210 (1999) 3–20.

[Grzegorczyk] Grzegorczyc, Andrzej: Some classes of recursive functions. *Rosprawy Math.*, 4 (1953) 1–46.

[Odifreddi] Odifreddi, Piergiorgio G.: *Classical recursion theory.* Studies in logic and the foundations of mathematics, vol. 125, North-Holand, 1989.

[Pour-El&Richards] Pour-El, M. B. & Rechards, J. I.: *Computability in Analysis and Physics.* Springer-Verlag, 1989.

[Weihrauch93] Weihrauch, Klaus: Computability on computable metric spaces. *Theoret. Comput. Sci.* 113 (1993) 191–210.

[Weihrauch00] Weihrauch, Klaus: Computable Analysis, Springer-verlag, 2000.

[YHK] Yamaguti, M., Hata, M. & Kigami, J.: *Hurakutaru no sûri* (in Japanese). Iwanami kôza ôyô sûgaku, Iwanami shoten, 1993.

This article was processed using the LaTeX macro package with LLNCS style