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Pruning Terms for Principal Type Assignment

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Abstract

Affine terms are lambda terms in which no variable occurs twice, and linear terms are affine terms in which each bound variable occurs exactly once. The principal type of a term is the most general type which is assignable to the term. Hirokawa gave characterisations of the principal types of linear terms and of beta-normal affine terms. Shouji gave a characterisation of the principal types of beta-eta-normal linear terms. The proofs of these three theorems are parallel, but independent to each other. This paper points out that by using Berardi's pruning, the theorem on beta-normal affine terms is obtained from the theorem on linear terms. This is the first result. And also this paper shows a characterisation of principal types of beta-eta-normal affine terms. Its proof is obtained from the theorem on beta-eta-normal linear terms by Berardi's pruning. This is the second result. Besides, this paper gives other proofs of the the theorems on eta-normal terms. This is the third result.

1 Introduction

1.1 Related Works and Motivation

The classes of linear terms and affine terms are subcategories of lambda terms. Affine terms are constructed such as no free variable appears both in M and N in application MN. Linear terms are constructed such as no free variable appears both in M and N in application MN, and moreover, the free variable x occurs exactly once in M in abstraction $\lambda x.M$.

Linear terms are isomorphic to proofs in natural deduction style of linear implicational logic, and affine terms are isomorphic to proofs in natural deduction style of affine implicational logic. Linear terms are often called BCI-lambda-terms, and linear implicational logic is often called BCI-logic because it is characterised by combinators B, C and I in the meaning of Hindley's theory [8]. Similarly, affine terms are often called BCK-lambda-terms, and affine implicational logic is often called BCK-logic.

For a type free term, the principal type is the most general type which can be assign to the term. Some studies of principal types were started with a pure interest of type assignment. Other studies of principal types were aroused with the studies of counting proofs of a formula. Hirokawa and Komori proved that in affine implicational logic, the number of normal proofs of a formula τ is equal to the number of minimal theorems τ' such that τ is an instance of τ' [6]. This holds also for linear implicational logic. Hirokawa proved that a formula is a minimal theorem of linear implicational logic iff it is the principal type of a $\beta\eta$-normal linear term [2]. He also proved the same property on affine terms [3]. Therefore, it is important to characterise principal types in linear implicational logic or affine implicational logic.

Hirokawa gave a characterisation of principal types of linear terms [2], and gave a characterisation of principal types of β-normal affine terms [5]. He also gave a conjecture of a characterisation of principal types of $\beta\eta$-normal linear terms [2], and Shouji proved the conjecture [9]. The characterisation of principal types of $\beta\eta$-normal affine terms is rested.

They proved these three theorems by using parallel heavy inductions. It is possible to prove the rest theorem on $\beta\eta$-normal affine terms by the similar induction. However, there are other ways to prove it.

Berardi proposed the method of pruning [7]. His pruning is a translation of explicitly typed λ-term into explicitly typed λI-terms, Explicitly typed terms are the terms in which each variable is explicitly typed in the application such as $\lambda x^T.M$, and λI-term are the terms in which the variable x occurs at least once freely in M in each abstraction $\lambda x.M$. The class of linear terms are a subcategory of λI-terms. The pruning reduces the problems on affine terms into those on linear terms. In this paper, we prove the characterisations of the principal types of affine terms from the characterisations of the principal types of linear terms by using Berardi's pruning.

Besides, we will show that by using the notion of minimality. One can prove the characterisations of $\beta\eta$-normal terms from the characterisations of β-normal terms and minimal types.

1.2 Main Results

The main results of this paper consist of three results.

The characterisation of principal types of linear terms by Hirokawa [2] is that a type τ is the principal type of a closed linear term iff the following three conditions hold.

(1) the type τ is balanced.
(2) the type τ is separable.
(3) the relevance graph $G(\tau)$ is a tree with $core(\tau)$ as its root.

And the characterisation of principal types of affine terms by Hirokawa [5] is that a type τ is the principal type of a closed β-normal affine term iff the following three conditions hold [5].

(1) the type τ is quasi-balanced.
(2) the type τ is separable.
(3) the relevance graph $G(\tau)$ consists of a tree with $core(\tau)$ as its root

and some isolated vertices.

The definition of balancedness is easy. If a type τ is balanced iff each atomic type in τ occurs exactly once positively and once negatively. If a type τ is quasi-balanced iff each atomic type in τ occurs exactly once positively and once negatively, or it occurs only once negatively. The definitions of relevanced graph and separability are a little bit complicated. They are defined from relevance relation, which denotes a relation of logical dependency between propositional letters in proving the formula, or in other words, dependency between atomic types in constructing a term of the type.

For each type τ, there are at most one closed β-normal affine term whose principal type is τ up to α-equivalence [4]. We write $Term(\tau)$ for such term. To say more precisely, the notation $Term(\tau)$ denotes some representative element of the set of closed β-normal affine terms whose principal type is τ. In this paper we identify α-equivalent terms, and we no more say that something is up to α-equivalence.

Let τ be a type satisfying the three conditions for affine terms. Put τ' as the type made from τ by erasing all the atomic types which occurs only negatively in τ. Then τ' satisfies the three conditions for linear terms. Our first result is the series of some lemmas on Berardi's pruning [7]. Our first result shows that $Term(\tau)$ is pruned into $Term(\tau')$.

It is easily seen that the characterisation on linear terms is obtained as an immediate consequence of that on affine terms, because the class of linear terms is a subcategory of affine terms. Our first result leads the inverse direction. We give another proof of the characterisation on affine terms as a corollary of the theorem on linear terms.

$$\begin{array}{ccc} & \text{an immediate consequence} & \\ \boxed{\begin{array}{c}\text{Hirokawa's Theorem}\\ \text{on linear terms}\end{array}} & \begin{array}{c}\longleftarrow\\ \longrightarrow\end{array} & \boxed{\begin{array}{c}\text{Hirokawa's theorem}\\ \text{on affine terms}\end{array}} \\ & \text{a corollary of our first result} & \end{array}$$

Shouji proved that a type τ is the principal type of a closed $\beta\eta$-normal linear term iff τ satisfies such three conditions for linear terms with the following additional condition [9].

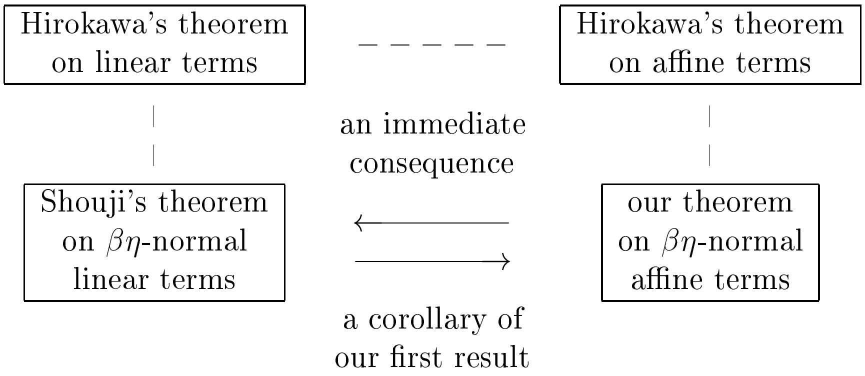
(4) τ satisfies subtype condition.

Subtype condition states that no subtype other than atomic types occurs twice. In other words, a type τ satisfies subtype condition iff it holds that if a type α occurs twice as the subtype in τ, then α is an atomic type. Shouji proved it by refining the induction steps of Hirokawa's theorem more precisely.

Our second result is the theorem on $\beta\eta$-normal affine terms, which is an extension of Shouji's theorem for affine terms. Our theorem states that a type τ is the principal type of a closed $\beta\eta$-normal affine term iff τ satisfies such three conditions for affine terms with the following additional condition.

(4) τ satisfies subtype condition.

Shouji's theorem is an immediate consequence of our second result. It is easy to prove our theorem directly with a little bit modification of Shouji's proof. But more interesting is to prove it by Shouji's theorem and Berardi's pruning. When we use Berardi's pruning, we can prove it without discussing the entangled induction steps around the conditions of tree-formed relevance graphs and separability.



Our third result is the vertical direction of the figure above. The definition of minimality is the followings. A type is linear minimal iff it is a locally most general type which is assigned to some closed linear term. A type is affine minimal iff it is a locally most general type which is assigned to some closed affine term. Hirokawa showed that a type is linear minimal iff it is the principal type of a closed $\beta\eta$-normal linear term [2], and that a type is affine minimal iff it is the principal type of a closed $\beta\eta$-normal affine term [3]. Suppose a type is the principal type of a term of a certain system. It is obvious that a type is minimal if it satisfies subtype condition. We show the inverse. If a type is linear minimal or affine minimal, then it satisfies subtype condition.

1.3 Outline

In Section 2, we define basic notions and show preliminary properties. In Section 3.1, we show Hirokawa's theorem on linear terms. In Section 3.2, we show Hirokawa's theorem on β-normal affine terms. In Section 4, we introduce Berardi's pruning, and explain our first result. In Section 5.1, we show Shouji's theorem on $\beta\eta$-normal linear terms. In Section 5.2, we show the theorem on $\beta\eta$-normal affine terms as our second result, and give a proof of our theorem by our first result. In Section 6, we show other proofs of the theorems on $\beta\eta$ -normal forms as our third result.

$$\begin{tikzcd}
 & \text{linear terms} & \text{affine terms} \\
\beta\text{-normal form} & \text{Theorem 3.1} \arrow[r, "\text{Proof in 4.3}"'] \arrow[d, "\text{Proof in 6.3}"'] & \text{Theorem 3.4} \arrow[d, "\text{Proof in 6.2}"] \\
\beta\eta\text{-normal form} & \text{Theorem 5.1} \arrow[r, "\text{Proof in 5.2}"'] & \text{Theorem 5.2}
\end{tikzcd}$$

2 Basic Notions and Preliminary Properties

2.1 Pure Type Assignment System

In this paper we deal with two systems, one is of linear terms and the other is of affine terms. Both are subsystems of Pure Type Assignment System. Pure Type Assignment System is one of most popular type assignment systems. Here we give the formal definition of it.

Definition 2.1 *Pure Type Assignment System* consists *types*, *untyped-terms* and *derivations* of type assignment. Untyped-terms play the role of pseudo-terms. An untyped-term are called a *term* when it has a derivations of type assignment. Types and untyped-terms are defined by the following syntax.

Atomic Type	$a, b, ...$	:	infinitely many
Type	$\tau, \upsilon, ...$	::=	$a \mid \tau \to \upsilon$
Variable	$x, y, ...$	:	infinitely many
Untyped-Term	$M, N, ...$	::=	$x \mid \lambda x.M \mid MN$

Derivations of type assignment are defined by the following three rules.

$$\text{Initial rule: } \frac{}{x:\tau \vdash x:\tau}\ , \quad \text{Introduction rule: } \frac{B \vdash M:\upsilon}{B - x:\tau \vdash \lambda x.M : \tau \to \upsilon}\ ,$$

$$\text{Elimination rule: } \frac{B \vdash M : \tau \to \upsilon \qquad B' \vdash N : \upsilon}{B, B' \vdash MN : \upsilon}\ .$$

The righthand of $\vdash$ consists of an untyped-term and a type connected with a colon. The lefthand of $\vdash$ is called a basis. A *basis* is consists of clauses $x : \tau$ where x is a variable and τ is a type Bases denote assignments of types to variables. Bases follow the following restriction. For a bases B, if two close $x : \tau$ and $x : \upsilon$ appears in B then $\tau = \upsilon$. This restriction prohibits a basis from assigning two types to a variable. The elimination rule applies only when the basis B, B' follows this restriction. The introduction rules applies only when the basis $B, x : \tau$ follows this restriction. In the introduction rule, the notation $B - x : \tau$ stands for the basis made form the basis B with deleting $x : \tau$ in B.

Definition 2.2 (Core) For a type τ, the *core* of the type τ, or $core(\tau)$, is the rightmost atomic type of a.

Definition 2.3 (Linear reduction) Let $M \to_\beta N$ be a step of β-reduction. Then the terms M and N have the forms such that $M = C[(\lambda x.X[x])Y]$ and $N = C[X[Y]]$. If the variable x occurs exactly once in $X[x]$, then we call the step $M \to_\beta N$ *linear-β-reduction.*

Proposition 2.4 (Subject expansion) *If $B \vdash M : \tau$ is derivable, and $N \to_\beta M$ is a step of linear-β-reduction, then $B \vdash N : \tau$ is also derivable.*

2.2 Linear Terms and Affine Terms

Here is the definitions of linear terms and of affine terms. Linear terms are called BCI-lambda-terms in Hirokawa's paper [2], and affine terms are called BCK-lambda-terms in Hirokawa's papers [5,3].

Definition 2.5 The system of linear terms is a subsystem of Pure Type Assignment System. *Linear Terms* are defined by the following syntax.

$$
\begin{array}{lll}
\textit{Term} \;\; M, N, ... ::= & x & \\
| & \lambda x.M & \text{where } x \in FV(M). \\
| & MN & \text{where } FV(M) \cup FV(N) = \emptyset.
\end{array}
$$

We call these terms *linear terms.* Derivations of type assignment are the derivations of Pure Type Assignment System which assigns types to only linear term.

Definition 2.6 The system of affine terms is a subsystem of Pure Type Assignment System. *Affine Terms* are defined by the following syntax.

$$
\begin{array}{ll}
\textit{Term} \;\; M, N, ... ::= & x \;\; | \;\; \lambda x.M \\
| & MN \quad \text{where } FV(M) \cup FV(N) = \emptyset.
\end{array}
$$

We call these terms *affine terms.* Derivations of type assignment are the derivations of Pure Type Assignment System which assigns types to only affine terms.

The class of linear terms and the class of affine terms are closed under β- and η-reduction. The class of linear terms is a subcategory of affine terms.

2.3 Principal Type

Suppose Pure Type Assignment System derives $B \vdash M : \tau$. Let θ be a substitution of atomic types with types, such as $[\tau_1/a_1, \tau_2/a_2, ...]$. Then it is easy to see that $B\theta \vdash M : \tau\theta$ is also derivable.

Definition 2.7 (Generality) Let τ and τ' be types. If there exists a substitution θ of atomic types with types and $\tau' = \tau\theta$, then τ is *more general* than τ'.

Let $B \vdash M : \tau$ and $B' \vdash M : \tau'$ be two judgments on the same term M of Pure Type Assignment System. If there exists a substitution θ of atomic types with types such that $B' = B\theta$ and $\tau' = \tau\theta$, then $B \vdash M : \tau$ is *more general* than $B' \vdash M : \tau'$.

Definition 2.8 (Principal Type) For a type M, a type τ and a basis B, if $B \vdash M : \tau$ is derivable, and for each derivable judgment $B' \vdash M : \tau'$, $B \vdash M : \tau$ is more general than $B' \vdash M : \tau'$, then $B \vdash M : \tau$ is called the *principal type assignment.* If the term M is closed, that is, the basis B is empty, then the type τ is called the *principal type* of M. When $B \vdash M : \tau$ is the principal type assignment, then we write $B \ \| \!\!- x : \tau$.

Obviously, for each term M, the principal type assignment is unique up to renaming atomic types, if it exists. We have the theorems on existence of the principal type assignments [1], and on uniqueness of the terms of principal types [4], which are the lemmas below.

Lemma 2.9 (Hindley) *Each β-normal affine term has its principal type assignment.*

Lemma 2.10 (Hirokawa) *If M and N are closed β-normal affine terms such that $\| \!\!- M : \tau$ and $\| \!\!- N : \tau$, then $M =_\alpha N$.*

We identify α-equivalent terms. Thus, we simply say for the previous theorem that for each type τ, there is at most one β-normal linear term whose principal type is τ if it exists.

Definition 2.11 (Subtype Condition) A type τ satisfies *subtype condition* iff no subtype of τ other than an atomic type occurs more than once in τ.

Proposition 2.12 *A type τ does not satisfy subtype condition iff there is a type υ and atomic types a, b, c such that a occurs at least twice in υ and $\tau = \upsilon[b \to c/a]$.*

Here the notation $[\alpha/a]$ denotes the substitution of a with α. A type $a \to a$ and a type $a \to b$ satisfy subtype condition, but not does a type $(a \to b) \to a \to b$.

2.4 Balanced Types and Quasi-Balanced Types

We define the notions of balanced types and quasi-balanced types.

Definition 2.13 (Balanced Type) A type τ is *balanced* iff each atomic type in τ occurs exactly once positively and exactly once negatively in τ.

Definition 2.14 (Quasi-Balanced Type) A type τ is *quasi-balanced* iff each atomic type in τ occurs exactly once negatively and never positively, or occurs exactly once negatively and exactly once negatively in τ.

A type $(a \to b) \to a \to b$ is balanced, hence quasi-balanced. A type $a \to b \to a$ is not balanced, but quasi-balanced. A type $(a \to a) \to b$ is not balanced, nor quasi-balanced.

2.5 Relevance Graph

Hirokawa defines the relevance relation and the relevance graph of a type [2].

Definition 2.15 (Relevance Relation) For a type τ, the *relevance relation* $\prec_\tau$ is a binary relation over atomic types in τ, which is defined as $a \prec_\tau b$ iff there is a negative subtype $\beta \to \alpha$ of τ such that $core(\alpha) = a$ and $core(\beta) = b$.

Definition 2.16 (Relevance Ordering) For a type τ, the relation $\prec^*_\tau$ is the transitive closure of $\prec_\tau$. The relation $\preceq^*_\tau$ is a preorder which is the reflexive closure of $\prec^*_\tau$.

The relation $\prec^*_\tau$ may not satisfy irrefrexivity, nor asymmetricity. For a type $\tau = (a \to b) \to (b \to a) \to c$, it is the case that $a \prec_\tau b$ and $b \prec_\tau a$, hence $a \prec^*_\tau a$.

Definition 2.17 (Relevance Graph) For a type τ, the relevance graph $G(\tau)$ is a directed graph with the atomic types in τ as its vertices. A directed graph means a graph where each edge has its source and destination. There is a edge from a to b in $G(\tau)$ iff the relevance relation $a \prec_\tau b$ holds.

Definition 2.18 (Tree) Suppose that $G = (V, E)$ be a directed graph, that is, the graph G has V as the set of its vertices and E as the set of its directed edges. A directed graph $G = (E, V)$ is a *tree* with its *root* r iff for each vertex $v \in V$, there exists exactly one sequence of vertices $\{r = v_1, v_2, ..., v_{n-1}, v_n = v\}$ such that $r = v_1 \prec v_2 \prec ... \prec v_{n-1} \prec v_n = v$, where $v \prec w$ stands for the edge from v to w in E. The length of the sequence n may be 1, thus, there are no edge destinated to the root r.

This definition provides the standard notion of trees.

Hirokawa did not use the notation of isolated vertices, but they simplify the statements of theorems. And he used the notions of separability, but he did not use this name.

Definition 2.19 (Isolated Vertex) Suppose that $G = (V, E)$ be a directed graph. A vertex $v \in V$ is *isolated* in G iff no edge starting from v, nor destinated to v.

Definition 2.20 (Separability) A type τ is *separable* iff for each subtype v of τ and for each a in v, either a is isolated in τ or $core(v) \preceq^*_\tau a$.

3 Theorems on Beta-Normal Form

In this section we show two theorems by Hirokawa. One is on linear terms, and the other is on β-normal affine terms,

3.1 Hirokawa's Theorem on Linear Terms

Hirokawa proves this theorem.

Theorem 3.1 (Hirokawa) *A type τ is the principal type of a closed linear term iff the type τ is balanced and separable, and $G(\tau)$ is a tree with $core(\tau)$ as its root.*

This theorem mentions all the closed linear terms, but it is enough to prove this theorem only on closed β-normal linear terms. That is because subject reduction and subject expansion (Lemma 2.4) hold for *beta*-reduction over linear terms. The proof appears his paper [2]. The proof consists of the following two lemmas.

Lemma 3.2 (Only-if part) *Let M be a β-normal linear term and $x_1 : \beta_1, ..., x_n : \beta_n \; ||\!\!- M : \tau$ be its principal type assignments. Then, the type $\tau' := \beta_1 \to \cdots \to \beta_n \to \tau$ is balanced and separable, and $G(\tau')$ is a tree with $core(\tau') = core(\tau)$ as its root.*

This is proved by induction on M.

Lemma 3.3 (If part) *If a type τ is balanced and separable, and the relevance graph $G(\tau)$ is a tree with $core(\tau)$ as its root. then there exists a closed β-normal linear term M such that $||\!\!- M : \tau$.*

This is proved by induction on τ. We write $Term(\tau)$ for this term M.

3.2 Hirokawa's Theorem on Affine Terms

In this section we show the generalisation of the theorem on linear terms into that on affine terms. The theorems on linear terms are obtained as the special case of the theorem in this section. The proof appears in Hirokawa's paper [5].

Theorem 3.4 (Hirokawa) *A type τ is the principal type of some closed β-normal affine term iff type is quasi-balanced and separable, and $G(\tau)$ consists of a tree with $core(\tau)$ as its root and some isolated vertices.*

In the statement, the notation of some isolated vertices means no isolated vertex or one isolated vertex or more isolated vertices. Indeed, if the affine term is linear, then $G(\tau)$ has no isolate vertex. The proof is similar to that on linear terms. The proof consists of the following two lemmas.

Lemma 3.5 (Only-if part) *Let M be a β-normal affine term, and $x_1 : \beta_1, ..., x_n : \beta_n \; ||\!\!- M : \tau$ be its principal type assignment. Then, the type $\tau' := \beta_1 \to \cdots \to \beta_n \to \tau$ is quasi-balanced and separable, and $G(\tau')$ consists of a tree with $core(\tau') = core(\tau)$ as its root and some isolated vertices.*

Lemma 3.6 (If part) *If a type τ is quasi-balanced and separable, and the relevance graph $G(\tau)$ consists a tree with $core(\tau)$ as its root and some isolated vertices. then there exists a closed β-normal affine term such that $||\!\!- M : \tau$.*

These lemmas are proved similarly to the case for linear terms. We also write $Term(\tau)$ for this term M. By reading the proofs of these lemmas in his paper [5] finely, we obtain other lemma and corollary.

Lemma 3.7 *Let M be a closed β-normal affine term and τ be the principal type of M. Then, the number of isolated vertices in $G(\tau)$ is equal to the number of subterms $\lambda x.L$ of M such that x does not appear in L.*

Corollary 3.8 *For a closed β-normal affine term M and its principal type τ, the term M is linear iff the type τ is balanced.*

4 Berardi's Pruning

In this section we discuss Berardi's pruning in his paper [7], which is a translation of affine terms and their principal types into linear terms and their principal types.

4.1 Explicitly Typed Terms

Here is the definition of Berardi's pruning operates upon explicitly typed terms.

Definition 4.1 *Explicitly typed terms* are defined similarly to the lambda terms with type assignment in Pure Type Assignment System. There are some points modified.

- There are special atomic type $\mathbf{U}$, which denotes unit, or the domain of nonsense data.
- There are an atomic constant () with the initial rule $\overline{\vdash () : \mathbf{U}}$.
- Each variable in terms consists of a variable name and a type superscript such as x^τ. And in type assignment, each basis assigns the same type τ for the variable x^τ.

We often omit the type superscripts of variables, that is, we simply write $xy : \upsilon$ instead of $x^{\tau\to\upsilon}y^\tau : \upsilon$.

The explicitly typed terms have the same information as the derivations of type assignment in pure type assignment system. Thus, from a derivation D in pure type assignment system, we can extract an explicitly typed term M with the same structure as D. We call the term M the explicitly typed term *extracted* from D.

By erasing all type superscripts in an explicitly typed terms M, we get an untyped-term N. We call N a *type free part* of M. We write M^{free} for the type free part of M.

Note that the notions of balanced types and quasi-balanced types are the notions over types of Pure Type Assignment System. Thus, no balanced type or no quasi-balanced type has $\mathbf{U}$.

4.2 Pruning

The essence of Berardi's pruning consists of the replacement of some types with the unit type $\mathbf{U}$ and the oblivion of some subterms of the type $\mathbf{U}$. The oblivion is formally defined as the following.

Definition 4.2 For each type τ and each term M of the system of explicitly typed terms, we define τ^{obl} and M^{obl} as:

$$
\begin{array}{rcll}
\mathbf{U}^{obl} &=& \mathbf{U} & \\
a^{obl} &=& a & \text{for an atomic type other than } \mathbf{U} \\
(\tau \to \upsilon)^{obl} &=& \upsilon^{obl} & \text{where } core(\tau) = \mathbf{U} \\
(\tau \to \upsilon)^{obl} &=& \tau^{obl} \to \upsilon^{obl} & \text{otherwise}
\end{array}
$$

$$
\begin{array}{rcll}
()^{obl} &=& () & \\
(x^{\tau})^{obl} &=& () & \text{where } core(\tau) = \mathbf{U} \\
(x^{\tau})^{obl} &=& x^{(\tau^{obl})} & \text{otherwise} \\
(\lambda x^{\tau}.M)^{obl} &=& M^{obl} & \text{where } core(\tau) = \mathbf{U} \\
(\lambda x^{\tau}.M)^{obl} &=& \lambda x^{(\tau^{obl})}.M^{obl} & \text{otherwise} \\
(MN)^{obl} &=& M^{obl} & \text{where } N^{obl} = () \\
(MN)^{obl} &=& M^{obl}N^{obl} & \text{otherwise}
\end{array}
$$

In the rest part of this section, we show our first result of this paper, which is the series of lemmas on pruning. We use the following assumption in some propositions below.

Condition 4.3 The type $\mathbf{U}$ occurs only negatively in τ. The type $\mathbf{U}$ may not occur in τ. The type $\mathbf{U}$ occurs only positively in υ but $core(\upsilon) \neq \mathbf{U}$. The type $\mathbf{U}$ may not occur in υ.

Definition 4.4 For a type τ and a term X, an untyped term $K_\tau[X]$ is defined as below.

$$
\begin{array}{rll}
K_\tau[X] = X & & \text{if } \tau \text{ does not have } \mathbf{U}. \\
K_\tau[X] \text{ undefined} & & \text{if } core(\tau) = \mathbf{U}. \\
K_{\upsilon\to\tau}[X] = \lambda x.K_\tau[X] & & \text{if } core(\tau) \neq \mathbf{U} \text{ and } core(\upsilon) = \mathbf{U}. \\
K_{\upsilon\to\tau}[X] = \lambda x.K_\tau[XK_\upsilon[x]] & & \text{if either } \tau \text{ or } \upsilon \text{ has } \mathbf{U} \text{ but } core(\tau) \neq \mathbf{U}.
\end{array}
$$

We write K_τ for $\lambda x.K_\tau[x]$

If $\tau \neq \mathbf{U}$, then K_τ is defined, and K_τ is closed $\beta\eta$-normal affine term.

Proposition 4.5 *Let τ and τ' be types such that $\tau = \tau'[\mathbf{U}/a_1, \mathbf{U}/a_2, ...]$ and τ' does not have $\mathbf{U}$.*

If $a_1, a_2, ...$ occur only negatively in τ, then the judgment $\vdash K_\tau : \tau^{obl} \to \tau'$ is derivable.

If $a_1, a_2, ...$ occur only positively in τ, and $core(\tau') \neq \mathbf{U}$, then the judgment $\vdash K_\tau : \tau' \to \tau^{obl}$ is derivable.

Proposition 4.6 *Suppose that types τ and υ satisfy the condition 4.3. Let M*

be a closed β-normal explicitly typed term. If $\vdash M : \tau$, *then* $K_\tau (M^{obl})^{free} \to_\beta^* M^{free}$. *If* $\vdash M : \upsilon$, *then* $K_\upsilon M^{free} \to_\beta^* (M^{obl})^{free}$.

Lemma 4.7 (Oblivion Lemma) *Let τ be a type which satisfies the condition 4.3, and M be a closed β-normal linear term. Put N as the β-normal form of $K_\tau M$. Then only linear β-reductions happen in the reduction sequence of $K_\tau M \to_\beta^* N$. And if M is η-normal, then so is N.*

Lemma 4.8 (Detachment Lemma) *Let M be an arbitrary closed β-normal affine term. Then there exist a type τ and a closed linear term N such that $M =_\beta K_\tau N$, and $\vdash M : \tau$.*

Lemma 4.9 (Principal Type Lemma) *Let τ and υ be types which satisfy the condition 4.3. Then* $\Vdash K_\tau : \psi(\tau) \to \phi(\tau)$, *and* $\Vdash K_\upsilon : \phi(\upsilon) \to \psi(\upsilon)$, *where $\phi(\ )$ and $\phi(\ )$ are defined as below.*

- $\phi(\alpha) = \psi(\alpha) = a$ *for a common fresh atomic type a, if α does not have* $\mathbf{U}$
- $\phi(\alpha \to \beta) = a \to \phi(\beta)$, $\psi(\alpha \to \beta) = \psi(\beta)$ *for a fresh atomic type a, if $core(\alpha) = \mathbf{U}$.*
- $\phi(\alpha \to \beta) = \phi(\alpha) \to \phi(\beta)$, $\psi(\alpha \to \beta) = \psi(\alpha) \to \psi(\beta)$ *if either α or β has $\mathbf{U}$, but $core(\alpha) \neq \mathbf{U}$. In this case, no atomic type appears both in $\phi(\alpha)$ and in $\phi(\beta)$.*

To say precisely, the notation ϕ and ψ do not denote functions. Neither $\phi(\tau)$ nor $\psi(\tau)$ is determined only with τ. They have varieties of fresh atomic types. Neither $\phi(\tau)$ nor $\psi(\tau)$ is determined by itself. They denote a ternary relation among τ, $\phi(\tau)$ and $\psi(\tau)$.

Lemma 4.10 (Preserving Lemma) *Let τ be a type, and $b_1, b_2, ..., b_n$ as the list of atomic types such that each of $b_1, b_2, ..., b_n$ occurs only negatively in τ, and is isolated in $G(\tau)$. Put θ as the substitution $[\mathbf{U}/b_1, \mathbf{U}/b_2, ..., \mathbf{U}/b_n]$, and τ' as $\tau' = (\tau\theta)^{obl}$. Then, the following conditions on τ and on τ' are equivalent to each other.*

- *The type τ is (1) balanced and (2) separable, and (3) the relevance graph $G(\tau)$ is a tree with $core(\tau)$ as its root.*
- *The type τ' is (1) quasi-balanced and (2) separable, and (3) the relevance graph $G(\tau')$ consists of a tree with $core(\tau')$ as its root and some isolate vertices.*

For example, a affine term $M = \lambda xy.x\lambda z.y$ is detached by the detachment lemma (4.8) into $M =_\beta K_{((\mathbf{U}\to a)\to b)\to a\to b} N$, where $N = \lambda xy.xy$.

The principal type assignment of M is

$$\Vdash \lambda xy.x\underline{\lambda z.}y : ((\underline{c \to} a) \to b) \to a \to b,$$

and the principal type assignment of N is

$$\Vdash \lambda xy.xy : (a \to b) \to a \to b,$$

which is obtained by pruning the underlined parts in the principal type assignment of M.

4.3 Another Proof of Hirokawa's Theorem

Here we show another proof of Theorem 3.4, which is Hirokawa's theorem on affine terms.

Proof of Lemma 3.5 (Only-if part of Theorem 3.4, the theorem on $\boldsymbol{\beta}$-normal affine terms), outline. Let M be a closed β-normal affine term. Then, by the detachment lemma (4.8), there is a type τ and a closed linear term N such that $K_\tau N =_\beta M$. By the lemma on linear terms (3.2), the principal type of N satisfies the tree conditions:

- It is balanced.
- It is separable.
- Its relevance graph is a tree with its core as the root.

Hence, by the preserving lemma (4.10), The principal type of M satisfies the three conditions in the lemma such as:

- It is quasi-balanced.
- It is separable.
- Its relevance graph consists of a tree with its core as the root and some isolated vertices.

□

Proof of Lemma 3.6 (If part of Theorem 3.4, the theorem on $\boldsymbol{\beta}$-normal affine terms), outline. Let τ be a type which satisfies the three conditions of the lemma such as:

- It is quasi-balanced.
- It is separable.
- Its relevance graph consists of a tree with its core as the root and some isolated vertices.

Put τ' as the type which is made by erasing all the atomic types which occurs only negatively in τ. So the type τ' satisfies these three conditions:

- It is balanced.
- It is separable.
- Its relevance graph is a tree with its core as the root.

Then, by the lemma on linear terms (3.3), there is a closed linear term N such that $\Vdash N : \tau'$. More over, there is a type υ such that $\Vdash K_\upsilon N : \tau$. Put M as the β-normal form of $K_\upsilon N$. Then M is a closed β-normal affine term. And that, by the oblivion lemma (4.7) and subject expansion (2.4), we have $\Vdash M : \tau$. □

5 Theorems on Eta-normal Form

5.1 Shouji's Theorem on Linear Terms

Hirokawa gave a conjecture in his paper [2]. Shouji prove it in his master thesis [9]. Shouji's proof of this theorem is similar to that of Hirokawa's theorem with some modification.

Theorem 5.1 (Shouji) *A type τ is a principal type of a $\beta\eta$-normal linear term iff τ is balanced and separable, and the relevance graph $G(\tau)$ is a tree with $core(\tau)$ as its root, and moreover, τ satisfies subtype condition.*

5.2 Theorem on affine terms

We prove the theorem on $\beta\eta$-normal affine terms.

Theorem 5.2 (Second Result) *A type τ is a principal type of a $\beta\eta$-normal affine term iff τ is quasi-balanced and separable, and the relevance graph $G(\tau)$ consists a tree with $core(\tau)$ as its root and some isolated vertices, and moreover, τ satisfies subtype condition.*

This theorem is directly obtained by a little bit modification of Shouji's proof. But we show another way of proof for each lemma, which is done by Shouji's theorem and some lemmas on pruning.

Lemma 5.3 (Subtype property after most general unification) *Let τ be a type in which* **U** *occurs only negatively. Put $\phi(\tau)$ and $\psi(\tau)$ such as the type $\psi(\tau) \to \phi(\tau)$ is a principal type of K_τ, as the principal type lemma (4.9). Let υ be a balanced type which satisfies subtype condition, and θ is a most general unifier of υ and $\psi(\tau)$. Then $\phi(\tau)\theta$ also satisfies subtype condition.*

Proof of Theorem 5.2. Similar to the proof in Section 4.3. We use Lemma 5.3 in order to show the subtype condition. □

6 Proofs from Beta-normal to Beta-eta-normal

In this section we show the proofs of the theorems on $\beta\eta$-normal terms by using the theorems on β-normal terms and the theorem on minimality. This is the third result of this paper.

6.1 Minimality

Definition 6.1 A type τ is *linear minimal* iff there is a linear term M such that $\vdash M : \tau$ and for each type τ' and linear term M', if $\vdash M' : \tau'$ and τ' is more general than τ, then τ is also more general than τ'.

Definition 6.2 A type τ is *affine minimal* iff there is an affine term M such that $\vdash M : \tau$ and for each type τ' and affine term M', if $\vdash M' : \tau'$ and τ' is more general than τ, then τ is also more general than τ'.

Hirokawa proved the following theorems. The proof for linear terms appears in his paper [3], and the proof for affine terms appears in his paper [2].

Theorem 6.3 (Hirokawa) *A type τ is linear minimal iff* $\| \!- M : \tau$ *for some $\beta\eta$-normal linear term M.*

Theorem 6.4 (Hirokawa) *A type τ is affine minimal iff* $\| \!- M : \tau$ *for some $\beta\eta$-normal affine term M.*

6.2 Another Proof of the Theorem on Affine Terms

Here is another proof of Shoji's theorem on $\beta\eta$-normal linear terms (5.1) by using the theorem on linear terms (3.1) and that on linear minimal types (6.3).

Proof of Theorem 5.2, outline. It is easy that a type τ which satisfies subtype condition cannot be a non-trivial instance of another type.

As the inverse, it is provable that if a type τ is the principal type of some closed β-normal affine term and it does not satisfy subtype condition, then the type τ is a non-trivial instance of another type τ' which is also the principal type of some affine term. Hence by *reductio ad absurdum* we have that if a type τ is the principal type of $\beta\eta$-normal affine term, then it is affine minimal by Theorem 6.4, thus it does satisfy subtype condition. □

6.3 Another Proof of the Theorem on Linear Terms

There is also another proof of our theorem on $\beta\eta$-normal affine terms (5.2).

Proof of Theorem 5.1, outline. The proof is done by using the theorem on linear terms (3.1) and that on linear minimal types (6.3), It is similar to the case for affine terms. □

7 Conclusion

We have prove the first results in Section 4.2, which characterise the term K_τ. As the consequence of that, we have given another proof of Hirokawa's theorem on β-normal affine terms (Theorem 3.4), and also we have proved the theorem on $\beta\eta$-normal affine terms (Theorem 5.2), which is the second result. Finally we gave other proofs of theorems on $\beta\eta$-normal linear terms (Theorem 5.1) on $\beta\eta$-normal affine terms (Theorem 5.2), as the third result.

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