## A Reconstruction of Ex Falso Quodlibet via Quasi-Multiple-Conclusion Natural Deduction

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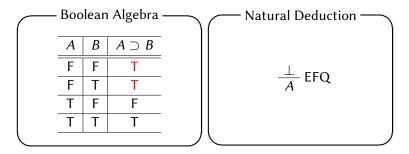
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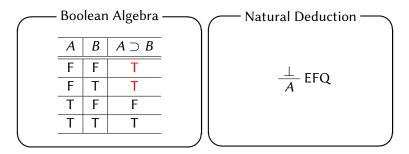
# *Ex Falso Quodlibet* (EFQ) is a logical principle which states that: "From the absurdity ' $\perp$ ', an arbitrary formula *A* follows"



It is now standard to view EFQ as a valid form of inference. However, how can we believe that this principle is "valid"? *Ex Falso Quodlibet* (EFQ) is a logical principle which states that: "From the absurdity ' $\perp$ ', an arbitrary formula *A* follows"



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- Aim To give an intuitive understanding of EFQ
- Idea To combine two ideas from logic and computation:
  - 1 " $^{\prime}$ ' is a logical dead-end in deductions" [Tennant '99]
  - 2 The catch/throw mechanism [Nakano '94]
- Result In our framework, EFQ can be viewed as:
  - "a jump inference from a dead-end to other possibilites"

1 Background: Meaning theory for formal logic

- 2 Several validations to EFQ and their problems
- 3 Our quasi-multiple-conclustion natural deduction: NQ
- **4** The corresponding typed  $\lambda$ -calculus:  $\lambda_{NQ}$

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<ロト < 団ト < 巨ト < 巨ト < 巨ト 三三 のへで 5 / 30 To define a formal deductive system, one has to consider that:

- "How to define mathematical proofs as mathematical entities?"
- "How can we believe that a mathematical inference is correct?"

## Meaning of mathematical statements



Michael Dummett

**C** The meaning of a mathematical statement determines and is exhaustively determined by its 'use'.

(The Philosophical Basis of Intuitionistic Logic, 1973)

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The meaning of inference rules of N.D. is determined by their "use"  
The conjuction rules as syntactic operations  

$$\frac{A \quad B}{A \land B} \land I \qquad \frac{A \land B}{A} \land E_L \qquad \frac{A \land B}{B} \land E_R$$

Its "use" lets us know the meaning: the commutativity of ' $\wedge$ ' –  $\frac{A \wedge B}{B} \wedge E_R \qquad \frac{A \wedge B}{A} \wedge E_L$   $\frac{B \wedge A}{B \wedge A} \wedge I$ 

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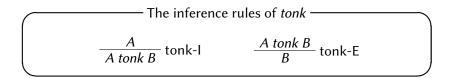
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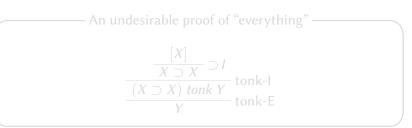
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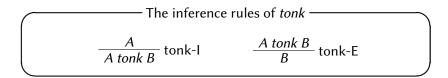
## Problematic connective: tonk [Prior '60]



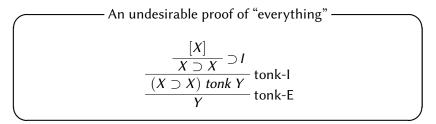
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- tonk-E is equivalent to  $\wedge E$



## Problematic connective: tonk [Prior '60]



- tonk-I is equivalent to  $\lor I$
- tonk-E is equivalent to  $\wedge E$





Gerhard Gentzen

**C** The introduction rules represent, as it were, the definition of the symbol concerned, and the elimination rules are no more, in the final analysis, than the consequences of these definitions.

"

(Investigations into Logical Inference, 1935)

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# Harmony Any consequence of a compound statement must be derived from its premises. Verified by *local reduction*

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Thanks to the harmony, we can say that tonk is invalid

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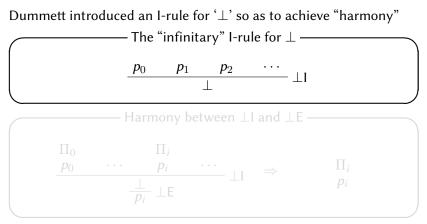
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Several validations to EFQ have been proposed so far:

- [Prawitz '07]: "There is no I-rule for '⊥' "
- Second-order prop. logic validation: " $\perp \stackrel{\text{def}}{=} \forall A.A$ "
- [Dummett '91]: By the harmonious I-rule for ' $\perp$ '
- Well-known validation: By the so-called *disjunctive syllogism* However, all of these validation have an inadequacy

# Dummett's Validation to EFQ [Dummett '91]

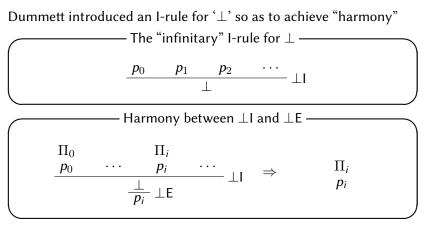


Problem:

- We have to admit such "infinitary" deduction;
- This does not explain why "'⊥' is what infers everything"

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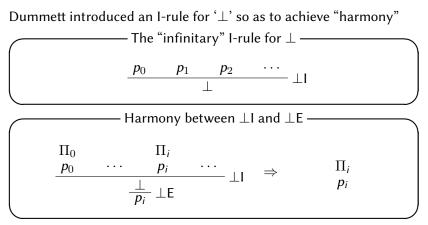


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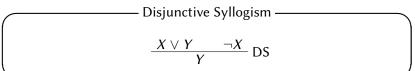


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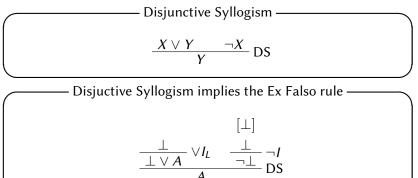
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According to [Priest '02], EFQ is historically validated by:



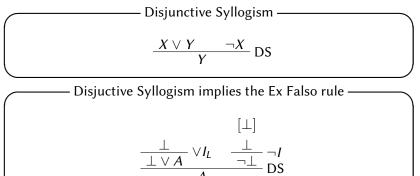
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#### Our obs.: EFQ is only used for "disjunctive syllogism"-like proofs

An informal proof of the disjunctive syllogism: " $X \lor Y, \neg X \vdash Y$ "

- 1 We have  $X \lor Y$ , that is, two possibilites that X or Y holds.
- 2 Firstly, focus on X. Considering that we also have  $\neg X$ , ' $\bot$ ' follows and hence this is not the case.
- <u>3</u> <u>Therefore</u>, Y holds.

This underlined inference is usually interpreted by depending on:

■ "'⊥' is what infers everything"

but an important point here is that:

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We introduce our intuitionistic natural deduction *NQ*, based on Nakano's constructive minimal logic [Nakano '94]

Definition (Formulae)

$$A, B ::= p \mid \bot \mid A \lor B \mid A \land B \mid A \supset B$$

#### Definition (Judgment [Nakano '94][Maehara '54])

A judgment of NQ, quasi-multiple-coclusion judgment, is a triple:

$$\Gamma \vdash A; \Delta$$

where A is a formula; each of  $\Gamma$  and  $\Delta$  is a multi-set of formulae

For a judgment  $\Gamma \vdash A$ ;  $\Delta$ , the conclusion "A;  $\Delta$ " expresses that:

- we are now "focusing" on a possibility A in reasoning proecss;
- $\blacksquare$  and "putting aside" the other possibilities to  $\Delta$

#### Remark: The "constructive" meaning of judgment

 $\Gamma \vdash A$ ;  $\Delta$  means that: "From the *constructions* of  $\Gamma$ , there exists at least one construction of the formula of  $\{A\} \cup \Delta$ "

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Following [Tennant '99]'s remark s.t. "' $^{\perp}$ ' is a logical dead-end", we formalize EFQ in our NQ by:

 $\frac{\Gamma \vdash \bot; A, \Delta}{\Gamma \vdash A; \Delta}$ 

Therefore EFQ in our formalization is ...

is no longer the inference deriving everything from ' $\perp$ '; but rather,

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## Example: A proof of disjunctive syllogism

#### Theorem (Disjunctive Syllogism)

 $\neg A, A \lor B \vdash B; \emptyset$  is derivable

#### Proof.

#### By the following derivation:

$$\frac{A \lor B \vdash A \lor B; \varnothing}{\neg A \vdash \neg A; \varnothing} Ax \qquad \underbrace{A \lor B \vdash A \lor B; \varnothing}_{A \lor B \vdash A; B} \lor E' \\ \xrightarrow{\neg A, A \lor B \vdash \bot; B}_{\neg A, A \lor B \vdash B; \varnothing} EFQ$$

### Our NQ logically corresponds to NJ as follows:

Theorem (Soundness and completeness w.r.t. NJ)

The followings are equivalent:

 $\begin{tabular}{ll} \begin{tabular}{ll} \Gamma \vdash_{NQ} A; \Delta \\ \begin{tabular}{ll} \begin{tabular}{ll} \Gamma \vdash_{NJ} \bigvee (\{A\} \cup \Delta), \end{tabular} \end{tabular} \end{tabular} \end{tabular}$ 

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## It is hard to say that some system is good in a general way, but Dummett proposed a criterion, called *harmony* [Dummett '91]

The notion of harmony is roughly summarized as follows:

- Reducibility of proof detour
- Normalizability of deduction

and these can be captured by the Curry–Howard correspondence

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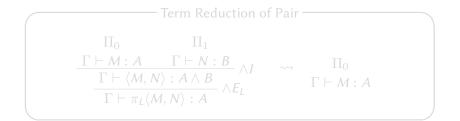
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The Curry-Howard correspondence [Curry '34][Howard '69] is a general correspondence between logic and computation:

Deductive System	Computational Model
formula	type
derivation	well-typed term
proof normalization	term reduction



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# The Corresponding Typed $\lambda$ -Calculus: $\lambda_{NQ}$

We introduce our calculus,  $\lambda_{NQ}$ , an extension of [Nakano '94]

Definition (Term/Judgment)

 $\begin{array}{l} (\operatorname{Term})\mathcal{M}, \mathcal{N}, \mathcal{L} ::= x \mid \lambda x.\mathcal{M} \mid \mathcal{M}\mathcal{N} \mid \langle \mathcal{M}, \mathcal{N} \rangle \mid \pi_{\mathcal{L}}\mathcal{M} \mid \pi_{\mathcal{R}}\mathcal{M} \mid \iota_{\mathcal{L}}\mathcal{M} \mid \iota_{\mathcal{R}}\mathcal{M} \\ \mid \mathbf{case} \ \mathcal{M} \ \mathbf{of} \ [x]\mathcal{N} \ \mathbf{or} \ [y]\mathcal{L} \mid u \mid \mathbf{throw} \ u \ \mathcal{M} \mid \mathbf{catch} \ u \ \mathcal{M} \\ (\operatorname{Judg.}) \ \Gamma \vdash \mathcal{M} : \mathcal{A}; \Delta \end{array}$ 

#### **Reduction examples**

$$\Pi \ \pi_L \langle M, N \rangle \rightsquigarrow M$$

**2** catch 
$$u(\pi_L \langle M, (\text{throw } u N) \rangle) \rightsquigarrow N$$

#### Theorem (The Curry-Howard Correspondence)

$$\square \Gamma \vdash_{NQ} A; \Delta iff \Gamma \vdash_{\lambda_{NQ}} M : A; \Delta for some M$$

**2** NQ's proof normalization corresponds to  $\lambda_{NQ}$ 's reduction

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$$\frac{\prod_{1}}{\Gamma \vdash N^{A \lor B};} \frac{\overline{Y^{B} \vdash y^{B};}^{Ax}}{x^{A} \vdash x^{A};} Ax \frac{\overline{y^{B} \vdash y^{B};}^{Ax}}{y^{B} \vdash (\text{throw } u \ y)^{A}; u^{B}} \nabla E$$

$$\frac{\Gamma \vdash M^{\neg A};}{\Gamma' \vdash (\text{case } N \text{ of } [x]x \text{ or } [y](\text{throw } u \ y))^{A}; u^{B}} \nabla E$$

$$\frac{\Gamma, \Gamma' \vdash (M(\text{case } N \text{ of } [x]x \text{ or } [y](\text{throw } u \ y)))^{\perp}; u^{B}}{\Gamma, \Gamma' \vdash (\text{catch } u \ (M(\text{case } N \text{ of } [x]x \text{ or } [y](\text{throw } u \ y))))^{B};} EFQ$$

The computational behaviour can be described by case analysis: 1 If  $N \equiv \iota_R(L)$  s.t. *L*'s type is *B*:

> catch  $u (M(\text{case } (\iota_R(L)) \text{ of } [x]x \text{ or } [y](\text{throw } u y)))$   $\rightsquigarrow$  catch u (M(throw u L)) $\rightsquigarrow L$

2 If 
$$N \equiv \iota_L(L)$$
 s.t. *L*'s type is *A*:

catch  $u (M(\text{case } (\iota_L(L)) \text{ of } [x]x \text{ or } [y](\text{throw } u y)))$   $\rightsquigarrow$  catch u (ML) $\rightsquigarrow \dots (\text{stuck})$ 

### The followings gurantee the well-definedness of NQ

Theorem (Subject reduction)

If  $\Gamma \vdash M : A; \Delta$  and  $M \rightsquigarrow M'$ , then  $\Gamma \vdash M' : A; \Delta$ 

#### Theorem (Strong normalization)

For every typable term *M*, there is no infinite reduction sequence starting from *M* 

Thanks to the above theorems, we can finally achieve the harmony:

Corollary (Harmony)

All the logical rules of NQ are in "harmony"

# Conclusion and Future work

Conclusion: We have proposed NQ to reconstruct EFQ

- In our NQ, EFQ is no longer the inference deriving everything, but is a jump inference from '⊥' to the other possibilities
- The justification of NQ is given by  $\lambda_{NQ}$  so as to achieve harmony
- The work indicates the single-conclusion N.D. is not enough to capture the reasoning structures of I.L.

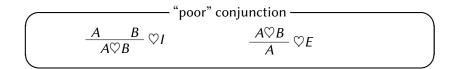
Future work: It is interesting as a further direction to ...

- To investigate a gap between our system and classical logic
- To investigate a relationship with relevance logic
- To establish a meaning theory through computational models

# Appendix

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# Another Problematic Connective: "poor" conjunction



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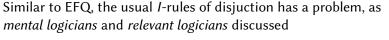
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- $\heartsuit I$  is equivalent to  $\land I$
- $\heartsuit E$  is equivalent to  $\land E_L$

## Stability The elimination rule must be as "strong" as possible. Achieved by *local expansion*.

Thanks to the stability, we can invalidate the poor conjunction

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– The disjunction rules produces an artbitary formula

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \lor B} \lor I_L \qquad \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \lor}$$

#### Theorem (Another formalization of disjunction)

*The following I- and E-rules for disjunction are derivable:* 

$$\frac{\Gamma \vdash A; B, \Delta}{\Gamma \vdash A \lor B; \Delta} \lor I' \qquad \qquad \frac{\Gamma \vdash A \lor B; \Delta}{\Gamma \vdash A; B, \Delta} \lor E'$$

By the following derivations, respectively:

$$\frac{\frac{\Gamma \vdash A; B, \Delta}{\Gamma \vdash A \lor B; B, \Delta} \lor I_{L}}{\frac{\Gamma \vdash B; A \lor B, \Delta}{\Gamma \vdash B; A \lor B; \Delta} \mathsf{Ex}} \xrightarrow{\Gamma \vdash A \lor B; \Delta} \frac{A \vdash A; \varnothing}{A \vdash A; \varnothing} \mathsf{Ax} \xrightarrow{B \vdash B; \varnothing} \mathsf{Ax} \mathsf{Throw}}{\frac{\Gamma \vdash A \lor B; A \lor B, \Delta}{\Gamma \vdash A \lor B; \Delta}} \mathsf{Catch} \xrightarrow{\Gamma \vdash A; B, \Delta} \mathsf{VE}$$

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# Inference rules of *NQ* and $\lambda_{NQ}$

