A Reconstruction of Ex Falso Quodlibet via Quasi-Multiple-Conclusion Natural Deduction

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Ex Falso Quodlibet

Ex Falso Quodlibet (EFQ) is a logical principle which states that:
"From the absurdity '$\bot$', an arbitrary formula A follows"

Boolean Algebra

A	B	$A \supset B$
F	F	T
F	T	T
T	F	F
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Natural Deduction

$$\frac{\bot}{A}\ \mathrm{EFQ}$$

It is now standard to view EFQ as a valid form of inference.
However, how can we believe that this principle is "valid"?

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Our work

Aim To give an intuitive understanding of EFQ

Idea To combine two ideas from logic and computation:

1. " '$\bot$' is a logical dead-end in deductions" [Tennant '99]
2. The catch/throw mechanism [Nakano '94]

Result In our framework, EFQ can be viewed as:
"a jump inference from a dead-end to other possibilites"

Outline

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How to define a formal deductive system?

To define a formal deductive system, one has to consider that:

- "How to define mathematical proofs as mathematical entities?"
- "How can we believe that a mathematical inference is correct?"

Michael Dummett

> “*The meaning of a mathematical statement determines and is exhaustively determined by its ‘use’.*”
>
> (*The Philosophical Basis of Intuitionistic Logic*, 1973)

One answer: Natural Deduction [Gentzen '34]

The meaning of inference rules of N.D. is determined by their "use"

The conjucntion rules as syntactic operations

$$\frac{A \qquad B}{A \land B} \land I \qquad \frac{A \land B}{A} \land E_L \qquad \frac{A \land B}{B} \land E_R$$

Its "use" lets us know the meaning: the commutativity of '$\land$'

$$\dfrac{\dfrac{A \land B}{B} \land E_R \qquad \dfrac{A \land B}{A} \land E_L}{B \land A} \land I$$

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Problematic connective: tonk [Prior '60]

The inference rules of *tonk*

$$\frac{A}{A \ tonk \ B}\ \text{tonk-I} \qquad \frac{A \ tonk \ B}{B}\ \text{tonk-E}$$

- tonk-I is equivalent to $\vee I$
- tonk-E is equivalent to $\wedge E$

An undesirable proof of "everything"

$$\dfrac{\dfrac{\dfrac{[X]}{X \supset X}\supset I}{(X \supset X)\ tonk\ Y}\ \text{tonk-I}}{Y}\ \text{tonk-E}$$

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Gentzen's explanatation of logical connectives

Gerhard Gentzen

> *"The introduction rules represent, as it were, the definition of the symbol concerned, and the elimination rules are no more, in the final analysis, than the consequences of these definitions."*

(*Investigations into Logical Inference*, 1935)

Harmony Any consequence of a compound statement must be derived from its premises. Verified by *local reduction*

$$\dfrac{\dfrac{\begin{array}{c}\Pi_0 \\ A\end{array} \quad \begin{array}{c}\Pi_1 \\ B\end{array}}{A \wedge B} \wedge I}{A} \wedge E_L \quad \rightarrow \quad \begin{array}{c}\Pi_0 \\ A\end{array}$$

Thanks to the harmony, we can say that *tonk* is invalid

Outline

Several previous validations to EFQ

Several validations to EFQ have been proposed so far:

- [Prawitz '07]: "There is no I-rule for '$\bot$' "
- Second-order prop. logic validation: "$\bot \stackrel{\text{def}}{=} \forall A.A$"
- [Dummett '91]: By the harmonious I-rule for '$\bot$'
- Well-known validation: By the so-called *disjunctive syllogism*

However, all of these validation have an inadequacy

Dummett's Validation to EFQ [Dummett '91]

Dummett introduced an I-rule for '⊥' so as to achieve "harmony"

The "infinitary" I-rule for ⊥

$$\frac{p_0 \qquad p_1 \qquad p_2 \qquad \cdots}{\bot}\ \bot\text{I}$$

Harmony between ⊥I and ⊥E

$$\dfrac{\dfrac{\begin{matrix}\Pi_0 \\ p_0\end{matrix} \qquad \cdots \qquad \begin{matrix}\Pi_i \\ p_i\end{matrix} \qquad \cdots}{\bot}\ \bot\text{I}}{p_i}\ \bot\text{E} \qquad \Rightarrow \qquad \begin{matrix}\Pi_i \\ p_i\end{matrix}$$

Problem:

- We have to admit such "infinitary" deduction;
- This does not explain why " '⊥' is what infers everything"

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Validation via Disjuctive Syllogism

According to [Priest '02], EFQ is historically validated by:

Disjunctive Syllogism

$$\frac{X \vee Y \qquad \neg X}{Y}\ \text{DS}$$

Disjuctive Syllogism implies the Ex Falso rule

$$\frac{\dfrac{\bot}{\bot \vee A}\ \vee I_L \qquad \dfrac{\begin{matrix}[\bot] \\ \bot\end{matrix}}{\neg\bot}\ \neg I}{A}\ \text{DS}$$

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Our key observation and how '$\bot$'/EFQ work

Our obs.: EFQ is only used for "disjunctive syllogism"-like proofs

An informal proof of the disjunctive syllogism: "$X \vee Y, \neg X \vdash Y$"

1. We have $X \vee Y$, that is, two possibilites that X or Y holds.
2. Firstly, focus on X. Considering that we also have $\neg X$, '$\bot$' follows and hence <u>this is not the case</u>.
3. <u>Therefore</u>, Y holds.

This underlined inference is usually interpreted by depending on:

- "'$\bot$' is what infers everything"

but an important point here is that:

- there seems to be a "global" relevance in the last two steps

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Our key observation and how '$\perp$'/EFQ work

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Our Quasi-Multiple-Conclusion Natural Deduction *NQ*

We introduce our intuitionistic natural deduction *NQ*, based on Nakano's constructive minimal logic [Nakano '94]

Definition (Formulae)

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \supset B$$

Definition (Judgment [Nakano '94][Maehara '54])

A judgment of *NQ*, *quasi-multiple-coclusion judgment*, is a triple:

$$\Gamma \vdash A; \Delta$$

where A is a formula; each of Γ and Δ is a multi-set of formulae

An intuitive meaning of judgment

For a judgment $\Gamma \vdash A; \Delta$, the conclusion "$A; \Delta$" expresses that:

- we are now "focusing" on a possibility A in reasoning proecss;
- and "putting aside" the other possibilities to Δ

Remark: The "constructive" meaning of judgment

$\Gamma \vdash A; \Delta$ means that: "From the *constructions* of Γ, there exists at least one construction of the formula of $\{A\} \cup \Delta$"

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EFQ as an "jump" inference

Following [Tennant '99]'s remark s.t. " '⊥' is a logical dead-end", we formalize EFQ in our *NQ* by:

$$\frac{\Gamma \vdash \bot; A, \Delta}{\Gamma \vdash A; \Delta}$$

Therefore EFQ in our formalization is ...

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Example: A proof of disjunctive syllogism

Theorem (Disjunctive Syllogism)

$\neg A, A \vee B \vdash B; \varnothing$ *is derivable*

Proof.

By the following derivation:

$$\dfrac{\dfrac{}{\neg A \vdash \neg A; \varnothing}\text{Ax} \qquad \dfrac{\dfrac{}{A \vee B \vdash A \vee B; \varnothing}\text{Ax}}{A \vee B \vdash A; B}\vee E'}{\dfrac{\neg A, A \vee B \vdash \bot; B}{\neg A, A \vee B \vdash B; \varnothing}\text{EFQ}}\supset E$$

□

Property of NQ

Our NQ logically corresponds to NJ as follows:

Theorem (Soundness and completeness w.r.t. NJ)

The followings are equivalent:

- $\Gamma \vdash_{NQ} A; \Delta$
- $\Gamma \vdash_{NJ} \bigvee(\{A\} \cup \Delta)$,

However, we can not yet conclude that NQ is a “good” system

Property of *NQ*

Our *NQ* logically corresponds to *NJ* as follows:

Theorem (Soundness and completeness w.r.t. *NJ*)

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What is a “good” deductive system?

It is hard to say that some system is good in a general way, but Dummett proposed a criterion, called *harmony* [Dummett ‘91]

The notion of harmony is roughly summarized as follows:

- Reducibility of proof detour
- Normalizability of deduction

and these can be captured by *the Curry–Howard correspondence*

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Background: The Curry–Howard correspondence

The Curry–Howard correspondence [Curry '34][Howard '69] is a general correspondence between logic and computation:

Deductive System	Computational Model
formula	type
derivation	well-typed term
proof normalization	term reduction

Term Reduction of Pair

$$\dfrac{\dfrac{\begin{matrix}\Pi_0 \\ \Gamma \vdash M : A\end{matrix} \qquad \begin{matrix}\Pi_1 \\ \Gamma \vdash N : B\end{matrix}}{\Gamma \vdash \langle M, N\rangle : A \wedge B} \wedge I}{\Gamma \vdash \pi_L \langle M, N\rangle : A} \wedge E_L \quad \rightsquigarrow \quad \begin{matrix}\Pi_0 \\ \Gamma \vdash M : A\end{matrix}$$

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The Corresponding Typed λ-Calculus: λ_{NQ}

We introduce our calculus, λ_{NQ}, an extension of [Nakano '94]

Definition (Term/Judgment)

(Term) $M, N, L ::= x \mid \lambda x.M \mid MN \mid \langle M, N \rangle \mid \pi_L M \mid \pi_R M \mid \iota_L M \mid \iota_R M$
$\mid \textbf{case } M \textbf{ of } [x]N \textbf{ or } [y]L \mid u \mid \textbf{throw } u\ M \mid \textbf{catch } u\ M$

(Judg.) $\Gamma \vdash M : A; \Delta$

Reduction examples

1. $\pi_L \langle M, N \rangle \rightsquigarrow M$
2. $\textbf{catch } u\ (\pi_L \langle M, (\textbf{throw } u\ N) \rangle) \rightsquigarrow N$

Theorem (The Curry–Howard Correspondence)

1. $\Gamma \vdash_{NQ} A; \Delta$ *iff* $\Gamma \vdash_{\lambda_{NQ}} M : A; \Delta$ *for some* M
2. *NQ's proof normalization corresponds to λ_{NQ}'s reduction*

Example: A Typing of Disjunctive Syllogism

$$
\dfrac{\dfrac{\begin{matrix}\Pi_0\\ \Gamma \vdash M^{\neg A};\end{matrix} \qquad \dfrac{\begin{matrix}\Pi_1\\ \Gamma' \vdash N^{A\vee B};\end{matrix} \quad \dfrac{}{x^A \vdash x^A;}\,\mathrm{Ax} \quad \dfrac{\dfrac{}{y^B \vdash y^B;}\,\mathrm{Ax}}{y^B \vdash (\mathbf{throw}\ u\ y)^A; u^B}\,\mathrm{Throw}}{\Gamma' \vdash (\mathbf{case}\ N\ \mathbf{of}\ [x]x\ \mathbf{or}\ [y](\mathbf{throw}\ u\ y))^A; u^B}\,\vee E}{\Gamma,\Gamma' \vdash (M(\mathbf{case}\ N\ \mathbf{of}\ [x]x\ \mathbf{or}\ [y](\mathbf{throw}\ u\ y)))^\perp; u^B}\,\supset E}{\Gamma,\Gamma' \vdash (\mathbf{catch}\ u\ (M(\mathbf{case}\ N\ \mathbf{of}\ [x]x\ \mathbf{or}\ [y](\mathbf{throw}\ u\ y))))^B;}\,\mathrm{EFQ}
$$

The computational behaviour of EFQ

The computational behaviour can be described by case analysis:

1. If $N \equiv \iota_R(L)$ s.t. L's type is B:

$$\begin{aligned} & \textbf{catch}\ u\ (M(\textbf{case}\ (\iota_R(L))\ \textbf{of}\ [x]x\ \textbf{or}\ [y](\textbf{throw}\ u\ y))) \\ & \rightsquigarrow \textbf{catch}\ u\ (M(\textbf{throw}\ u\ L)) \\ & \rightsquigarrow L \end{aligned}$$

2. If $N \equiv \iota_L(L)$ s.t. L's type is A:

$$\begin{aligned} & \textbf{catch}\ u\ (M(\textbf{case}\ (\iota_L(L))\ \textbf{of}\ [x]x\ \textbf{or}\ [y](\textbf{throw}\ u\ y))) \\ & \rightsquigarrow \textbf{catch}\ u\ (ML) \\ & \rightsquigarrow \ldots\ (\text{stuck}) \end{aligned}$$

Properties

The followings gurantee the well-definedness of *NQ*

Theorem (Subject reduction)

If $\Gamma \vdash M : A; \Delta$ *and* $M \rightsquigarrow M'$, *then* $\Gamma \vdash M' : A; \Delta$

Theorem (Strong normalization)

For every typable term M, there is no infinite reduction sequence starting from M

Thanks to the above theorems, we can finally achieve the harmony:

Corollary (Harmony)

All the logical rules of NQ are in "harmony"

Conclusion and Future work

- Conclusion: We have proposed *NQ* to reconstruct EFQ
  - In our *NQ*, EFQ is no longer the inference deriving everything, but is a jump inference from '$\perp$' to the other possibilities
  - The justification of *NQ* is given by λ_{NQ} so as to achieve harmony
  - The work indicates the single-conclusion N.D. is not enough to capture the reasoning structures of I.L.

- Future work: It is interesting as a further direction to ...
  - To investigate a gap between our system and classical logic
  - To investigate a relationship with relevance logic
  - To establish a meaning theory through computational models

Appendix

Another Problematic Connective: "poor" conjunction

"poor" conjunction

$$\frac{A \qquad B}{A \heartsuit B}\,\heartsuit I \qquad\qquad \frac{A \heartsuit B}{A}\,\heartsuit E$$

- $\heartsuit I$ is equivalent to $\wedge I$
- $\heartsuit E$ is equivalent to $\wedge E_L$

Stability [Dummett '91][Pfenning&Davies '00]

Stability The elimination rule must be as "strong" as possible. Achieved by *local expansion.*

$$\begin{array}{c}\Pi\\ A \land B\end{array} \quad \to \quad \dfrac{\dfrac{\begin{array}{c}\Pi\\ A \land B\end{array}}{A}\land E_L \qquad \dfrac{\begin{array}{c}\Pi\\ A \land B\end{array}}{B}\land E_R}{A \land B}\land I$$

Thanks to the stability, we can invalidate the poor conjunction

A problem of disjunction rules

Similar to EFQ, the usual *I*-rules of disjuction has a problem, as *mental logicians* and *relevant logicians* discussed

The disjunction rules produces an artbitary formula

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee I_L \qquad\qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee I_R$$

Example: Reconstruction of disjunction

Theorem (Another formalization of disjunction)

The following I- and E-rules for disjunction are derivable:

$$\frac{\Gamma \vdash A; B, \Delta}{\Gamma \vdash A \vee B; \Delta} \vee I' \qquad\qquad \frac{\Gamma \vdash A \vee B; \Delta}{\Gamma \vdash A; B, \Delta} \vee E'$$

By the following derivations, respectively:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash A; B, \Delta}{\Gamma \vdash A \vee B; B, \Delta} \vee I_L}{\Gamma \vdash B; A \vee B, \Delta} \text{Ex}}{\Gamma \vdash A \vee B; A \vee B, \Delta} \vee I_R}{\Gamma \vdash A \vee B; \Delta} \text{Catch} \qquad \dfrac{\Gamma \vdash A \vee B; \Delta \quad \dfrac{}{A \vdash A; \varnothing} \text{Ax} \quad \dfrac{\dfrac{}{B \vdash B; \varnothing} \text{Ax}}{B \vdash A; B} \text{Throw}}{\Gamma \vdash A; B, \Delta} \vee E$$

where Ex is a derivable rule

Inference rules of NQ and λ_{NQ}

NQ's inference rules

$$\frac{\Gamma \vdash \bot; A, \Delta}{\Gamma \vdash A; \Delta}\ \text{EFQ} \qquad \frac{\Gamma \vdash A; A, \Delta}{\Gamma \vdash A; \Delta}\ \text{Catch} \qquad \frac{\Gamma \vdash A; \Delta}{\Gamma \vdash B; A, \Delta}\ \text{Throw}$$

λ_{NQ}'s typing rules

$$\frac{\Gamma \vdash M : \bot; u : A, \Delta}{\Gamma \vdash \mathbf{catch}\ u\ M : A; \Delta}\ \text{EFQ}$$

$$\frac{\Gamma \vdash M : A; u : A, \Delta}{\Gamma \vdash \mathbf{catch}\ u\ M : A; \Delta}\ \text{Catch}$$

$$\frac{\Gamma \vdash M : A; \Delta}{\Gamma \vdash \mathbf{throw}\ u\ M : B; u : A, \Delta}\ \text{Throw}$$