On a computational interpretation of sequent calculus for modal logic S4

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Modal logic in the theory of programming languages

Some studies [Kobayashi '97][Benton+ '98][Pfenning+ '01][Kimura+ '11] discovered that S4 corresponds to various typed λ-calculi for "meta-programming"

In the logical foundation, $\Box$-modality plays an essential role:

- $\Box A$ means the set of programs which "encode" programs of type A
- (this is similar to the intuition in Logic of Proof, etc: $\Box A$ means the proposition of a "proof" of A)

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Problem from a practical viewpoint

All the previous studies only consider natural-deduction-style λ-calculi, and they use the "one-step" substitution as usual:

$$(\lambda x.M)\,N \rightsquigarrow M[x := N]$$

However, the operation is too rich from a practical viewpoint

Natural-deduction-style λ-calculus is not enough to capture the structure of computation

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Aim of this talk

To create another computational model for modal logic S4, in terms of sequent calculus

To do this, a sequent calculus and its corresponding calculus for intuitionistic S4 are proposed

1. proof-theoretically based on:
   - a modal sequent calculus and the G3-style system [Troelstra&Schwichtenberg '96]
   - a higher-arity modal natural deduction [Pfenning&Davies '01]
2. type-theoretically based on:
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Higher-arity Sequent Calculus for intuitionistic S4

We propose a "higher-arity" sequent calc. for $(\wedge, \vee, \supset, \Box)$-fragment of intuitionistic S4, $\mathbf{HLJ_{S4}}$, based on [Troelstra&Schwichtenberg '96]

Definition (Formula)

$$A, B ::= p \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A$$

Definition (Higher-arity judgment [Pfenning+ '01])

A *judgment* is defined by the following higher-arity form:

$$\Delta ; \Gamma \vdash A$$

which intuitively means $(\bigwedge \Box \Delta) \wedge (\bigwedge \Gamma) \supset A$

Inference rules of $\mathbf{HLJ_{S4}}$

$$\frac{}{\emptyset; A \vdash A}\ \mathrm{Ax} \qquad \frac{}{A; \emptyset \vdash A}\ \Box\mathrm{Ax}$$

$$\frac{\Delta; \Gamma \vdash A \qquad \Delta; \Gamma \vdash B}{\Delta; \Gamma \vdash A \wedge B}\ \wedge R \qquad \frac{\Delta; \Gamma, A_i \vdash B}{\Delta; \Gamma, A_1 \wedge A_2 \vdash B}\ \wedge L$$

$$\frac{\Delta; \Gamma \vdash A_i}{\Delta; \Gamma \vdash A_1 \vee A_2}\ \vee R \qquad \frac{\Delta; \Gamma, A \vdash C \qquad \Delta; \Gamma, B \vdash C}{\Delta; \Gamma, A \vee B \vdash C}\ \vee L$$

$$\frac{\Delta; \Gamma, A \vdash B}{\Delta; \Gamma \vdash A \supset B}\ \supset R \qquad \frac{\Delta; \Gamma \vdash A \qquad \Delta; \Gamma, B \vdash C}{\Delta; \Gamma, A \supset B \vdash C}\ \supset L$$

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$$\frac{\Delta; \Gamma \vdash B}{\Delta; \Gamma, A \vdash B}\ \mathrm{W} \qquad \frac{\Delta; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B}\ \Box\mathrm{W}$$

$$\frac{\Delta; \Gamma, A, A \vdash B}{\Delta; \Gamma, A \vdash B}\ \mathrm{C} \qquad \frac{\Delta, A, A; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B}\ \Box\mathrm{C}$$

$$\frac{\Delta; \Gamma \vdash A \qquad \Delta; \Gamma, A \vdash B}{\Delta; \Gamma \vdash B}\ \mathrm{Cut} \qquad \frac{\Delta; \emptyset \vdash A \qquad \Delta, A; \Gamma \vdash B}{\Delta; \Gamma \vdash B}\ \Box\mathrm{Cut}$$

Init rules of $\mathbf{HLJ_{S4}}$

Init rule

$$\frac{}{\emptyset; A \vdash A}\ \text{Ax} \qquad \frac{}{A; \emptyset \vdash A}\ \Box\text{Ax}$$

Intuition of Ax

$$\frac{}{\Box A \supset A}\ \Box\text{Ax}$$

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Logical rules of $\mathbf{HLJ_{S4}}$

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$$\frac{\Delta; \Gamma \vdash A \qquad \Delta; \Gamma \vdash B}{\Delta; \Gamma \vdash A \wedge B} \wedge R \qquad \frac{\Delta; \Gamma, A_i \vdash B}{\Delta; \Gamma, A_1 \wedge A_2 \vdash B} \wedge L$$

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Structural rules of $\mathbf{HLJ_{S4}}$

Structural rule

$$\frac{\Delta; \Gamma \vdash B}{\Delta; \Gamma, A \vdash B}\ \mathrm{W} \qquad \frac{\Delta; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B}\ \Box\mathrm{W}$$

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Cut rule

$$\frac{\Delta; \Gamma \vdash A \qquad \Delta; \Gamma, A \vdash B}{\Delta; \Gamma \vdash B}\ \mathrm{Cut} \qquad \frac{\Delta; \emptyset \vdash A \qquad \Delta, A; \Gamma \vdash B}{\Delta; \Gamma \vdash B}\ \Box\mathrm{Cut}$$

On the cut-elimination procedure

While we can prove the cut-elimination theorem for $\mathbf{HLJ_{S4}}$, the proof by the mix-elimination is problematic; because ...

$$\dfrac{\begin{array}{c}\Pi \\ \Delta;\Gamma \vdash A\end{array} \quad \dfrac{\begin{array}{c}\Pi' \\ \Delta';\Gamma',A,A \vdash B\end{array}}{\Delta';\Gamma',A \vdash B}\,\mathrm{C}}{\Delta,\Delta';\Gamma,\Gamma' \vdash B}\,\mathrm{Cut} \implies \dfrac{\begin{array}{c}\Pi \\ \Delta;\Gamma \vdash A\end{array} \quad \begin{array}{c}\Pi' \\ \Delta';\Gamma',A,A \vdash B\end{array}}{\Delta,\Delta';\Gamma,\Gamma' \vdash B}\,\mathrm{Mix}$$

In the elimination procedure,

- it is "okay" if we consider the provability of the judgment; but
- it is "not okay" if we consider the construction of the judgment

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G3-style sequent calculus

The *G3-style* [Kleene '52][Dragalin '88] is a style of formalization to make a cut-free, or precisely, "structural-rule-free" system

The G3-style inference rules are defined in a somewhat tricky way to derive the "height-preserving admissible" structural rules

G3-style system for $\mathbf{HLJ_{S4}}$, named $\mathbf{G3\text{-}HLJ_{S4}}$

The G3-style inference rules are defined as follows:

$$\frac{}{\Delta; \Gamma, A \vdash A}\ \mathrm{Ax} \qquad \frac{}{\Delta, A; \Gamma \vdash A}\ \Box\mathrm{Ax}$$

$$\frac{\Delta; \Gamma, A \vdash B}{\Delta; \Gamma \vdash A \supset B}\ {\supset} R \qquad \frac{\Delta; \Gamma, A \supset B \vdash A \qquad \Delta; \Gamma, A \supset B, B \vdash C}{\Delta; \Gamma, A \supset B \vdash C}\ {\supset} L$$

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From the original rules to the G3-style

Idea: all we have to do is to get "height-preserving" structural rules

Original $\mathbf{HLJ_{S4}}$

$$\frac{}{\emptyset; A \vdash A}\,\mathrm{Ax}$$

$$\frac{\Delta; \Gamma, A_i \vdash B}{\Delta; \Gamma, A_1 \wedge A_2 \vdash B}\,\wedge L$$

$$\frac{\Delta, A; \Gamma \vdash B}{\Delta; \Gamma, \Box A \vdash B}\,\Box L$$

$\Longrightarrow$

G3-style $\mathbf{G3\text{-}HLJ_{S4}}$

$$\frac{}{\Delta; \Gamma, A \vdash A}\,\mathrm{Ax}$$

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Desired properties

Lemma (Height-preserving weakening/contraction)

The followings are height-preserving admissible rules in $\mathbf{G3\text{-}HLJ_{S4}}$*:*

$$\frac{\Delta; \Gamma \vdash B}{\Delta; \Gamma, A \vdash B}\, W \qquad \frac{\Delta; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B}\, \Box W$$

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Theorem (Equivalence)

The provability of $\mathbf{HLJ_{S4}}$ *and* $\mathbf{G3\text{-}HLJ_{S4}} + \mathbf{Cut}$ *is equivalent*

Theorem (Cut-elimination)

The cut rules Cut and $\Box$*Cut are admissible in* $\mathbf{G3\text{-}HLJ_{S4}}$

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Term assignment for the modal sequent calculus

We propose a term assignment system for the **G3-HLJ$_{\mathbf{S4}}$**, $\lambda^{\Box}_{seq}$, to get the computational model

As [Ohori '99] did for a G3-style prop. int. sequent calc., we assign terms to **G3-HLJ$_{\mathbf{S4}}$** + **Cut** as follows:

- Init/Right rules: assign λ-terms, as we do for N.D. system
- Left/Cut rules: assign the so-called "let expression"

Good point: $\lambda^{\Box}_{seq}$ does not use "meta-level" substitution!

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Term assignment for init/right rules

Assign the modal λ-term [Pfenning+ '01] to the init/right rules:

$$\frac{}{\Delta; \Gamma, x : A \vdash x : A}\ \mathsf{Ax} \qquad \frac{}{\Delta, u : A; \Gamma \vdash u : A}\ \Box\mathsf{Ax}$$

$$\frac{\Delta; \Gamma \vdash M : A \qquad \Delta; \Gamma \vdash N : B}{\Delta; \Gamma \vdash \langle M, N \rangle : A \wedge B}\ \wedge R$$

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$$\frac{\Delta; \emptyset \vdash M : A}{\Delta; \Gamma \vdash \mathbf{box}\, M : \Box A}\ \Box R$$

Term assignment for left rule of conjunction

Assign "let-expression" to the left conjunction rule:

$$\frac{\Delta; \Gamma, x : A \wedge B, y : A, z : B \vdash M : C}{\Delta; \Gamma, x : A \wedge B \vdash \mathbf{let}\, \langle y, z \rangle = x \,\mathbf{in}\, M : C} \wedge L$$

The reduction intuitively proceeeds, e.g., as:

$$(\mathbf{let}\, \langle y, z \rangle = \langle N, L \rangle \,\mathbf{in}\, M) \rightsquigarrow M[y := N, z := L]$$

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Term assignment for the other left rules

The rules for the other left rules are defined similarly:

$$\frac{\Delta; \Gamma, x : A \supset B \vdash M : A \qquad \Delta; \Gamma, x : A \supset B, y : B \vdash N : C}{\Delta; \Gamma, x : A \supset B \vdash \textbf{let}\, y = x\, M\, \textbf{in}\, N : C} \supset L$$

$$\frac{\Delta, u : A; \Gamma, x : \Box A \vdash M : B}{\Delta; \Gamma, x : \Box A \vdash \textbf{let box}\, u = x\, \textbf{in}\, M : B} \Box L$$

Term assignment for cut rules

The term assignment for cut rules are defined as a "composition" of two constructions, again by using let-expressions:

$$\frac{\Delta; \Gamma \vdash M : A \qquad \Delta; \Gamma, x : A \vdash N : B}{\Delta; \Gamma \vdash \textbf{let}\, x = M\, \textbf{in}\, N : B}\ \text{Cut}$$

$$\frac{\Delta; \emptyset \vdash M : A \qquad \Delta, u : A; \Gamma \vdash N : B}{\Delta; \Gamma \vdash \textbf{let}\, u = M\, \textbf{in}\, N : B}\ \Box\text{Cut}$$

Cut-elimination in terms of $\lambda^{\Box}_{seq}$

Let us consider the cut-elimination for conjunction:

$$\dfrac{\dfrac{\vdash M : A \qquad \vdash N : B}{\vdash \langle M, N\rangle : A \wedge B}\wedge R \qquad \dfrac{x : A \wedge B, y : A, z : B \vdash L : C}{x : A \wedge B \vdash \mathbf{let}\,\langle y, z\rangle = x\,\mathbf{in}\,L : C}\wedge L}{\vdash \mathbf{let}\,x = \langle M, N\rangle\,\mathbf{in}\,\mathbf{let}\,\langle y, z\rangle = x\,\mathbf{in}\,L : C}\text{Cut}$$

To eliminate cuts, all we have to do is to compute:

$$(\mathbf{let}\,x = \langle M, N\rangle\,\mathbf{in}\,\mathbf{let}\,\langle y, z\rangle = x\,\mathbf{in}\,L) \rightsquigarrow L[y := M, z := N, x := \langle M, N\rangle]$$

but we do not want to use "meta-level" substitution

Fortunatelly, the (local) cut-elimination step defined in the G3-style is exactly what we want!

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To eliminate cuts, all we have to do is to compute:

$$\begin{gathered}(\mathbf{let}\,x = \langle M, N\rangle\,\mathbf{in}\,\mathbf{let}\,\langle y, z\rangle = x\,\mathbf{in}\,L)\\ \rightsquigarrow L[y := M, z := N, x := \langle M, N\rangle]\end{gathered}$$

but we do not want to use “meta-level” substitution

Fortunatelly, the (local) cut-elimination step defined in the G3-style is exactly what we want!

Local cut-elimination as program-simplification

(A part of) translation rules are obtained as follows:

Optimization

$$(\mathbf{let}\, x = M\, \mathbf{in}\, x) \rightsquigarrow M$$

$$(\mathbf{let}\, x = M\, \mathbf{in}\, y) \rightsquigarrow y$$

Flattening

$$(\mathbf{let}\, w = (\mathbf{let}\, \langle y, z\rangle = x\, \mathbf{in}\, M)\, \mathbf{in}\, N) \rightsquigarrow (\mathbf{let}\, \langle y, z\rangle = x\, \mathbf{in}\, \mathbf{let}\, w = M\, \mathbf{in}\, N)$$

$$(\mathbf{let}\, y = (\mathbf{let}\, \mathbf{box}\, u = x\, \mathbf{in}\, M)\, \mathbf{in}\, N) \rightsquigarrow (\mathbf{let}\, \mathbf{box}\, u = x\, \mathbf{in}\, \mathbf{let}\, y = M\, \mathbf{in}\, N)$$

Decomposition

$$(\mathbf{let}\, x = \langle M, N\rangle\, \mathbf{in}\, \mathbf{let}\, \langle y, z\rangle = x\, \mathbf{in}\, L) \rightsquigarrow (\mathbf{let}\, y = M\, \mathbf{in}\, \mathbf{let}\, z = N\, \mathbf{in}\, \mathbf{let}\, x = \langle y, z\rangle\, \mathbf{in}\, L)$$

$$(\mathbf{let}\, x = \mathbf{box}\, M\, \mathbf{in}\, \mathbf{let}\, \mathbf{box}\, u = x\, \mathbf{in}\, N) \rightsquigarrow (\mathbf{let}\, u = M\, \mathbf{in}\, \mathbf{let}\, x = \mathbf{box}\, u\, \mathbf{in}\, N)$$

These translation corresponds to "A-normal form compilation" in the theory of programming languages

Local cut-elimination as program-simplification

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Decomposition

$$(\mathbf{let}\, x = \langle M, N\rangle\, \mathbf{in}\, \mathbf{let}\, \langle y, z\rangle = x\, \mathbf{in}\, L) \rightsquigarrow (\mathbf{let}\, y = M\, \mathbf{in}\, \mathbf{let}\, z = N\, \mathbf{in}\, \mathbf{let}\, x = \langle y, z\rangle\, \mathbf{in}\, L)$$
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These translation corresponds to "A-normal form compilation" in the theory of programming languages

Properties of $\lambda_{seq}^{\Box}$ and the cut-elimination theorem

Theorem (Subject reduction)

If $\Delta; \Gamma \vdash M : A$ and $M \rightsquigarrow M'$, then $\Delta; \Gamma \vdash M' : A$

Theorem (Strong normalization)

Every typable term is strongly normalizing

Corollary (Cut-elimination theorem)

$\lambda_{seq}^{\Box}$ enjoys the cut-elimination theorem, which also yields that every typable term can be reduced to the unique normal form

The following tells us that $\lambda^{\Box}_{seq}$ can be used as a basis of model for the existing theory:

Theorem (Embedding from modal typed λ-calculus)

The modal λ-calc. $\lambda^{\Box}$ [Pfenning+ '01] can be embeded into $\lambda^{\Box}_{seq}$:

- *If* $\Delta; \Gamma \vdash M : A$ *in* $\lambda^{\Box}$, *then* $\Delta; \Gamma \vdash [\![M]\!] : A$ *in* $\lambda^{\Box}_{seq}$
- *If* $M \rightsquigarrow M'$ *in* $\lambda^{\Box}$, *then* $[\![M]\!] \rightsquigarrow [\![M']\!]$ *in* $\lambda^{\Box}_{seq}$

where $[\![-]\!]$ *means the translation mapping from* $\lambda^{\Box}$ *to* $\lambda^{\Box}_{seq}$

Conclusion and future work

- Conclusion
  - A cut-free higher-arity sequent calc. for intuitionistic S4: $\mathbf{HLJ_{S4}}$ and $\mathbf{G3\text{-}HLJ_{S4}}$
  - (A cut-free higher-arity sequent calc. for classical S4: $\mathbf{HLK_{S4}}$ and $\mathbf{G3\text{-}HLK_{S4}}$)
  - The corresponding term calculus for $\mathbf{G3\text{-}HLJ_{S4}}$
- Future work
  - The corresponding term calculus for the classical version, following the work of $\lambda\mu$-calculus for modal logic [Kimura+ '11]
  - (Ongoing work with Akira Yoshimizu): Geometry of Interaction semantics for modal logic in terms of MELL, following the work of GoI semantics for PCF [Mackie '95]

Appendix

Cut-elimination (1)

$$\begin{aligned}
\textbf{let}\ x = y\ \textbf{in}\ M &\rightsquigarrow M[x := y] \\
\textbf{let}\ x = u\ \textbf{in}\ M &\rightsquigarrow M[x := u] \\
\textbf{let}\ u = v\ \textbf{in}\ M &\rightsquigarrow M[u := v] \\
\textbf{let}\ x = M\ \textbf{in}\ x &\rightsquigarrow M \\
\textbf{let}\ x = M\ \textbf{in}\ y &\rightsquigarrow y \\
\textbf{let}\ u = M\ \textbf{in}\ x &\rightsquigarrow x \\
\textbf{let}\ u = M\ \textbf{in}\ u &\rightsquigarrow M \\
\textbf{let}\ u = M\ \textbf{in}\ v &\rightsquigarrow v \\
\textbf{let}\ x = M\ \textbf{in}\ u &\rightsquigarrow u \\
\textbf{let}\ z = (\textbf{let}\ y = x\,M\ \textbf{in}\ N)\ \textbf{in}\ L &\rightsquigarrow \textbf{let}\ y = x\,M\ \textbf{in}\ \textbf{let}\ z = N\ \textbf{in}\ L \\
\textbf{let}\ w = (\textbf{let}\ \langle y, z\rangle = x\ \textbf{in}\ M)\ \textbf{in}\ N &\rightsquigarrow \textbf{let}\ \langle y, z\rangle = x\ \textbf{in}\ \textbf{let}\ w = M\ \textbf{in}\ N \\
\textbf{let}\ w = (\textbf{case}\ x\ \textbf{of}\ [y]M\ \textbf{or}\ [z]N)\ \textbf{in}\ L &\rightsquigarrow \textbf{case}\ x\ \textbf{of}\ [y](\textbf{let}\ w = M\ \textbf{in}\ L)\ \textbf{or}\ [z](\textbf{let}\ w = N\ \textbf{in}\ L) \\
\textbf{let}\ y = (\textbf{let box}\ u = x\ \textbf{in}\ M)\ \textbf{in}\ N &\rightsquigarrow \textbf{let box}\ u = x\ \textbf{in}\ \textbf{let}\ y = M\ \textbf{in}\ N
\end{aligned}$$

Cut-elimination (2)

$$
\begin{aligned}
\textbf{let}\ x = L\ \textbf{in let}\ z = y\ M\ \textbf{in}\ N &\rightsquigarrow \textbf{let}\ z = y\ (\textbf{let}\ x = L\ \textbf{in}\ M)\ \textbf{in let}\ x = L\ \textbf{in}\ N \\
\textbf{let}\ x = N\ \textbf{in let}\ \langle y, z\rangle = w\ \textbf{in}\ M &\rightsquigarrow \textbf{let}\ \langle y, z\rangle = w\ \textbf{in let}\ x = N\ \textbf{in}\ M \\
\textbf{let}\ x = L\ \textbf{in case}\ w\ \textbf{of}\ [y]M\ \textbf{or}\ [z]N &\rightsquigarrow \textbf{case}\ w\ \textbf{of}\ [y](\textbf{let}\ x = L\ \textbf{in}\ M)\ \textbf{or}\ [z](\textbf{let}\ x = L\ \textbf{in}\ N) \\
\textbf{let}\ x = N\ \textbf{in let box}\ u = y\ \textbf{in}\ M &\rightsquigarrow \textbf{let box}\ u = y\ \textbf{in let}\ x = N\ \textbf{in}\ M \\
\textbf{let}\ y = \lambda x : A.M\ \textbf{in let}\ z = y\ N\ \textbf{in}\ L &\rightsquigarrow \textbf{let}\ y = \lambda x : A.M\ \textbf{in let}\ x = N\ \textbf{in let}\ z = M\ \textbf{in}\ L \\
\textbf{let}\ x = \langle M, N\rangle\ \textbf{in let}\ \langle y, z\rangle = x\ \textbf{in}\ L &\rightsquigarrow \textbf{let}\ y = M\ \textbf{in let}\ z = N\ \textbf{in let}\ x = \langle y, z\rangle\ \textbf{in}\ L \\
\textbf{let}\ x = \iota_l^{A\vee B}(M)\ \textbf{in case}\ x\ \textbf{of}\ [y]N\ \textbf{or}\ [z]L &\rightsquigarrow \textbf{let}\ y = M\ \textbf{in let}\ x = \iota_l^{A\vee B}(y)\ \textbf{in}\ N \\
\textbf{let}\ x = \iota_r^{A\vee B}(M)\ \textbf{in case}\ x\ \textbf{of}\ [y]N\ \textbf{or}\ [z]L &\rightsquigarrow \textbf{let}\ z = M\ \textbf{in let}\ x = \iota_r^{A\vee B}(z)\ \textbf{in}\ L \\
\textbf{let}\ x = \textbf{box}\ M\ \textbf{in let box}\ u = x\ \textbf{in}\ N &\rightsquigarrow \textbf{let}\ u = M\ \textbf{in let}\ x = \textbf{box}\ u\ \textbf{in}\ N
\end{aligned}
$$