A Linear-Logical Reconstruction of Intuitionistic Modal Logic S4

Yosuke Fukuda[1] Akira Yoshimizu[2]

[1]Graduate School of Informatics, Kyoto University, Japan

[2]INRIA, France

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Linear-logical reconstruction of intuitionistic logic

The *Girard translation* $(-)^\circ$ allows us to reconstruct intuitionistic logic in terms of linear logic, decomposing $\supset$ to $\multimap$, ! [Girard '87]:

$$(p)^\circ \stackrel{\mathrm{def}}{=} p \qquad\qquad (p : \text{atomic})$$
$$(A \supset B)^\circ \stackrel{\mathrm{def}}{=} (!(A)^\circ) \multimap (B)^\circ$$

Soundness of the Girard translation

If $\Gamma \vdash A$ in intuitionistic logic, then $!(\Gamma)^\circ \vdash (A)^\circ$ in linear logic, where $!(\Gamma)^\circ \stackrel{\mathrm{def}}{=} \{!(A)^\circ \mid A \in \Gamma\}$

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The Girard translation under the Curry–Howard

The Curry–Howard correspondence tells us that:
the Girard translation is also "correct" w.r.t. proof-normalizations

Simply-typed λ-calc., $\lambda^{\supset}$ ($\supset$-fragment of int. logic)

$$\Gamma \vdash M : A$$

- Γ : intuitionistic context

$\xrightarrow{(-)^{\circ}}$

A λ-calc. of DILL [Barber '96] ((!, $\multimap$)-fragment of linear logic)

$$\Gamma ; \Sigma \vdash M : A$$

- Γ : intuitionistic context
- Σ : linear context

Soundness of the Girard translation $(-)^{\circ}$

1. If $\Gamma \vdash M : A$ in $\lambda^{\supset}$, then $(\Gamma)^{\circ} ; \emptyset \vdash (M)^{\circ} : (A)^{\circ}$ in DILL
2. If $M \rightsquigarrow M'$ in $\lambda^{\supset}$, then $(M)^{\circ} \rightsquigarrow (M')^{\circ}$ in DILL

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Our work: linear-logical reconstruciton of modal logic

Motivation
To give computational interpretations for various intuitionistic modal logics by linear logic (w/ Geometry of Interaction semantics)

This talk A linear-logical reconstruction of the $(\Box, \supset)$-fragment of intuitionistic S4, and its computational interpretations

Contribution

1. Modal linear logic, an integration of modal logic & linear logic
2. Typed λ-calculus for the modal linear logic
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1. Naïve attempt at “modal linear logic”
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What should “the modal linear logic” be?

Ordinary Girard trans.

$$\begin{array}{c}\vdots\\ \Gamma \vdash A\end{array} \quad \overset{(-)^\circ}{\longmapsto} \quad \begin{array}{c}\vdots\\ !(\Gamma)^\circ \vdash (A)^\circ\end{array}$$

in intuitionistic logic with $\supset$

in (intuitionistic) linear logic with $(!, \multimap)$

S4 Girard trans.

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(Something corresponding to the S4 derivation)

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A modal linear logic(?)

We review a naïve combination of modal logic and linear logic, $\mathrm{IMELL}^{\Box}$, so as to be a target logic of an S4 Girard translation

Syntax

$$\text{Formula} \quad A, B ::= p \mid A \multimap B \mid !A \mid \Box A$$

Inference rules

$$\frac{}{A \vdash A}\ \mathrm{Ax} \qquad \frac{\Gamma \vdash A \quad A, \Gamma' \vdash B}{\Gamma, \Gamma' \vdash B}\ \mathrm{Cut}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ {\multimap}\mathrm{R} \qquad \frac{\Gamma \vdash A \quad \Gamma', B \vdash C}{\Gamma, \Gamma', A \multimap B \vdash C}\ {\multimap}\mathrm{L}$$

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Naïve Girard translation from Intuitionistic S4

Naïve Girard trans. $(-)^\circ$: Intuitionistic S4 $\to \mathrm{IMELL}^\Box$ is ...

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Goal If $\Gamma \vdash A$ in Int. S4, then $!(\Gamma)^\circ \vdash (A)^\circ$ in $\mathrm{IMELL}^\Box$.

Fact The above statement of soundness is invalid!

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Problematic case

Goal (recall) If $\Gamma \vdash A$ in Int. S4, then $!(\Gamma)^\circ \vdash (A)^\circ$ in $\mathrm{IMELL}^\Box$.

Problematic case in the translation

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in Int. S4 — in $\mathrm{IMELL}^\Box$

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The problem of the previous inference:

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$$\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ !\mathrm{R} \qquad \dfrac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\ \Box\mathrm{R}$$

Remark There also exists a counter-example even if we use
$(\Box A)^\circ \stackrel{\mathrm{def}}{=} !\Box(A)^\circ$ or $(\Box A)^\circ \stackrel{\mathrm{def}}{=} \Box!(A)^\circ$

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Modal linear logic

Modal linear logic, called $\mathrm{IMELL}^{\boxed{!}}$, is defined to be an extension of the $(!, \multimap)$-fragment of intuitionistic linear logic with $\boxed{!}$-modality

Syntax

$$\text{Formula} \quad A, B ::= p \mid A \multimap B \mid !A \mid \boxed{!}A$$

Intuition of ! and $\boxed{!}$

- ! admits the structural rules of weakening and contraction
- $\boxed{!}$ is an integration of ! and $\Box$, meaning that:
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Rules for weakening/contraction/dereliction

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where $\delta \in \{!, \boxed{!}\}$

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S4-version of Girard translation

Definition (S4 Girard translation)

$(-)^\circ$: Int. S4 formulae $\to \mathrm{IMELL}^{\boxdot}$ formulae is defined as follows:

$$(p)^\circ \stackrel{\text{def}}{=} p$$

$$(A \supset B)^\circ \stackrel{\text{def}}{=} !(A)^\circ \multimap (B)^\circ$$

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Theorem (Soundness of $(-)^\circ$ w.r.t. provability)

If $\Box\Gamma, \Gamma' \vdash A$ in Int. S4, then $\boxdot(\Gamma)^\circ, !(\Gamma')^\circ \vdash A$ in $\mathrm{IMELL}^{\boxdot}$.

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If $\Box\Gamma, \Gamma' \vdash A$ *in Int. S4, then* $\boxed{!}(\Gamma)^\circ, !(\Gamma')^\circ \vdash A$ *in* $\mathrm{IMELL}^{\boxed{!}}$.

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Typed λ-calculus for modal linear logic

The typed λ-calculus, called $\lambda^{\boxed{!}}$, is defined as an integration of

1. the λ-calculus for intuitionistic S4 [Pfenning&Davies '00, '01]
2. the λ-calculus for dual intuitionistic linear logic [Barber '96]

Syntax

Type $A, B ::= p \mid A \multimap B \mid\ !A \mid \boxed{!}A$

Term $M, N ::= x \mid \lambda x : A.M \mid M\,N$

$\mid\ !M \mid \textbf{let}\ !x = M\ \textbf{in}\ N$

$\mid \boxed{!}M \mid \textbf{let}\ \boxed{!}x = M\ \textbf{in}\ N$

Reduction

$$(\lambda x : A.M)\,N \rightsquigarrow M[N/x]$$

$$\textbf{let}\ !x =\ !N\ \textbf{in}\ M \rightsquigarrow M[N/x]$$

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Type judgment

$$\Delta; \Gamma; \Sigma \vdash M : A$$

where Δ, Γ, Σ are multi-sets of formulae, and

1. Δ implicitly represents a context for types of form $\boxed{!}A$;
2. Γ implicitly represents a context for types of form $\Box A$;
3. Σ represents an ordinary context but is used linearly.

Typed λ-calculus for modal linear logic

The typed λ-calculus, called $\lambda^{\boxed{!}}$, is defined as an integration of

1. the λ-calculus for intuitionistic S4 [Pfenning&Davies '00, '01]
2. the λ-calculus for dual intuitionistic linear logic [Barber '96]

Syntax

Type $A, B ::= p \mid A \multimap B \mid\ !A \mid \boxed{!}A$

Term $M, N ::= x \mid \lambda x : A.M \mid M\,N$
$\mid\ !M \mid \textbf{let}\ !x = M\ \textbf{in}\ N$
$\mid \boxed{!}M \mid \textbf{let}\ \boxed{!}x = M\ \textbf{in}\ N$

Reduction

$$(\lambda x : A.M)\,N \rightsquigarrow M[N/x]$$
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Reduction

$$(\lambda x : A.M)\,N \leadsto M[N/x]$$
$$\textbf{let}\ !x = !N\ \textbf{in}\ M \leadsto M[N/x]$$
$$\textbf{let}\ \boxed{!}x = \boxed{!}N\ \textbf{in}\ M \leadsto M[N/x]$$

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Several typing rules in $\lambda^{\boxed{!}}$

Rules for $\multimap$

$$\frac{\Delta; \Gamma; \Sigma, x : A \vdash M : B}{\Delta; \Gamma; \Sigma \vdash \lambda x : A.M : A \multimap B} \multimap \mathrm{I}$$

$$\frac{\Delta; \Gamma; \Sigma \vdash M : A \multimap B \qquad \Delta; \Gamma; \Sigma' \vdash N : A}{\Delta; \Gamma; \Sigma, \Sigma' \vdash M\,N : B} \multimap \mathrm{E}$$

Rules for ! and $\boxed{!}$

$$\frac{\Delta; \Gamma; \emptyset \vdash M : A}{\Delta; \Gamma; \emptyset \vdash !M : !A} !\mathrm{I} \qquad \frac{\Delta; \emptyset; \emptyset \vdash M : A}{\Delta; \Gamma; \emptyset \vdash \boxed{!}M : \boxed{!}A} \boxed{!}\mathrm{I}$$

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(Note: $!\mathrm{E}$ is defined similarly to $\boxed{!}\mathrm{E}$)

Fact If $\boxed{!}\Delta, !\Gamma, \Sigma \vdash A$ is derivable in $\mathrm{IMELL}^{\boxed{!}}$, then $\Delta; \Gamma; \Sigma \vdash M : A$ is also derivable in $\lambda^{\boxed{!}}$ for some M

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Rules for $\multimap$

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Rules for $!$ and $\boxed{!}$

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S4 Girard translation à la λ-calc.

$\lambda^{\Box}$ [Davies&Pfenning '01]
Intuitionistic S4 w/ ($\Box$, $\supset$)

$$\Delta; \Gamma \vdash M : A$$

- Δ : modal context
- Γ : int. context

$\overset{(-)^\circ}{\longmapsto}$

Our $\lambda^{\boxed{!}}$ w/ ($\boxed{!}$, !, $\multimap$)

$$\Delta; \Gamma; \Sigma \vdash M : A$$

- Δ : modal ($\boxed{!}$) context
- Γ : int. (!) context
- Σ : linear context

Term $(M)^\circ$: $\lambda^{\Box}$-terms $\to$ $\lambda^{\boxed{!}}$-terms

$$(x)^\circ \overset{\text{def}}{=} x$$

$$(\lambda x : A.M)^\circ \overset{\text{def}}{=} \lambda y :!(A)^\circ.\textbf{let}\, !x = y\, \textbf{in}\, (M)^\circ$$

$$(M\, N)^\circ \overset{\text{def}}{=} (M)^\circ\, !(N)^\circ$$

$$(\Box M)^\circ \overset{\text{def}}{=} \boxed{!}(M)^\circ$$

$$(\textbf{let}\, \Box x = M\, \textbf{in}\, N)^\circ \overset{\text{def}}{=} \textbf{let}\, \boxed{!}x = (M)^\circ\, \textbf{in}\, (N)^\circ$$

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Soundness of the S4 Girard translation

Theorem (Soundness of $(-)^\circ$)

1. *If $\Delta; \Gamma \vdash M : A$ is derivable in $\lambda^{\Box}$, then $(\Delta)^\circ; (\Gamma)^\circ; \emptyset \vdash (M)^\circ : (A)^\circ$ is also derivable in $\lambda^{\boxed{!}}$.*
2. *If $M \rightsquigarrow M'$ in $\lambda^{\Box}$, then $(M)^\circ \rightsquigarrow^+ (M')^\circ$ in $\lambda^{\boxed{!}}$.*

The formalizations of modal linear logic and the above tells us that:

Modal logic can be interpreted in linear logic!

Soundness of the S4 Girard translation

Theorem (Soundness of $(-)^\circ$)

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Outline

1. Naïve attempt at "modal linear logic"
2. (Intuitionistic) modal linear logic
3. Typed λ-calculus for modal linear logic
4. Geometry of Interaction semantics for modal linear logic

Formalization of the GoI semantics

We follow the work of the so-called *GoI Machine* [Mackie '94, '95] to give a GoI semantics for our $\lambda^{\boxed{!}}$

Steps of the construction

1. A sequent calculus for classical modal linear logic
2. A proof-net formalization for the classical sequent calculus
3. A reduction-preserving embedding from $\lambda^{\boxed{!}}$ to proof-nets
4. An extended dynamic algbera with the notion of path
5. A particle-style (a.k.a token-passing-style) GoI semantics

Main theorem $\lambda^{\boxed{!}}$ (and hence $\lambda^{\Box}$) can be interpreted by the GoI

Important point: Thanks to "the simplicity" of our logic, we can obtain it just as a straightforward extension of existing work

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Classical sequent calculus

A sequent calc. for modal linear logic, $\mathrm{CMELL}^{\boxed{!}}$, is defined as:

Syntax

$$\text{Formula } A, B ::= p \mid p^{\perp} \mid A \otimes B \mid A \parr B \mid\, !A \mid\, ?A \mid \boxed{!}A \mid \Diamond\!?\,A$$

(where $A \multimap B$ is defined as $A^{\perp} \parr B$)

with the equations of dual formula, e.g., $(\boxed{!}A)^{\perp} = \Diamond\!?\,(A^{\perp})$

(A part of) rules

$$\frac{\vdash \Diamond\!?\,\Delta, ?\Gamma, A}{\vdash \Diamond\!?\,\Delta, ?\Gamma, !A}\ ! \qquad \frac{\vdash \Diamond\!?\,\Delta, A}{\vdash \Diamond\!?\,\Delta, \boxed{!}A}\ \boxed{!}$$

Recall In $\mathrm{IMELL}^{\boxed{!}}$,

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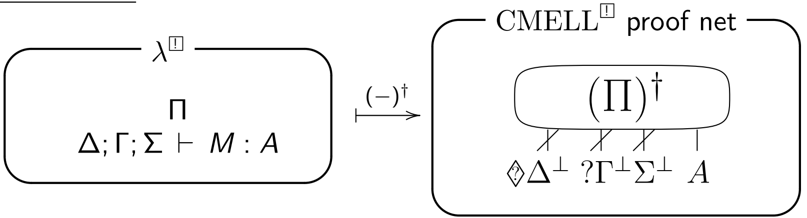
Embedding from $\lambda^{\Box}$ to CMELL proof nets

Embedding



Example

Embedding from $\lambda^{\Box}$ to CMELL proof nets

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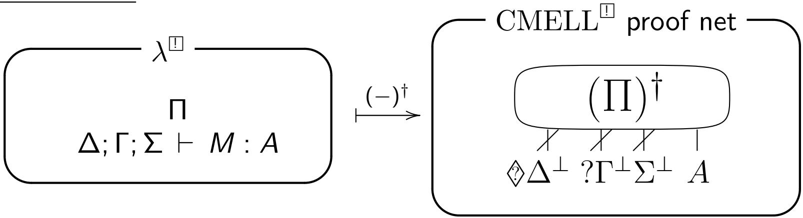
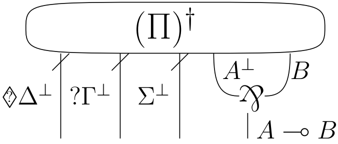


Example

$$\frac{\begin{array}{c}\Pi\\ \Delta;\Gamma;\Sigma,x:A\vdash M:B\end{array}}{\Delta;\Gamma;\Sigma\vdash \lambda x:A.M:A\multimap B}\multimap\mathrm{I}$$



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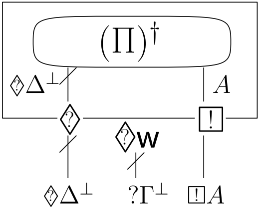

GoI interpretaiton of CMELL proof nets

1. An extended dynamic algebra $\Lambda^{\Box *}$, a single-sorted Σ algebra
   - Constants $0, 1, p, q, r, r', s, s', t, t', d, d' : \Sigma$
   - Operators $(\cdot) : \Sigma \times \Sigma \to \Sigma$, $! : \Sigma \to \Sigma$, $\boxed{!} : \Sigma \to \Sigma$
   - (with several conditions to define "good" proof-nets)
2. Algebraic characterization of nets (with the notion of *path*)
3. A GoI (Machine) interpretation à la context semantics (with the notion of *execution formula*)
   - The computation is characterized by "token-traversing" of path, using a *context*, an intermediate state of an abstract machine

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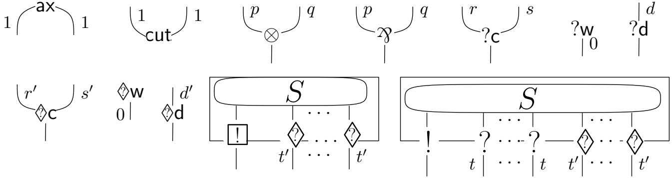

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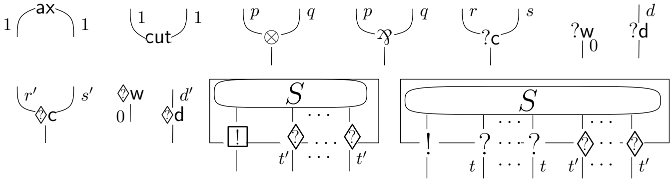
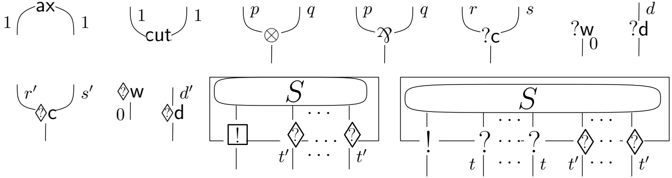


3. A GoI (Machine) interpretation à la context semantics (with the notion of *execution formula*)
   - The computation is characterized by "token-traversing" of path, using a *context*, an intermediate state of an abstract machine

GoI interpretaiton of CMELL proof nets

1. An extended dynamic algebra $\Lambda^{\Box *}$, a single-sorted Σ algebra
   - Constants $0, 1, p, q, r, r', s, s', t, t', d, d' : \Sigma$
   - Operators $(\cdot) : \Sigma \times \Sigma \to \Sigma$, $! : \Sigma \to \Sigma$, $\boxed{!} : \Sigma \to \Sigma$
   - (with several conditions to define "good" proof-nets)
2. Algebraic characterization of nets (with the notion of *path*)



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Properties on the GoI interpretation

Lemma

Let $\mathcal{N}$ be a closed proof net and $\mathcal{N}'$ be its normal form. Then, $[\![\mathcal{N}]\!] = [\![\mathcal{N}']\!]$, where $[\![-]\!]$ returns the denotation by the GoI

Theorem (Soundness of the GoI interpretation of $\lambda^{\boxdot}$)

For a closed well-typed term M in $\lambda^{\boxdot}$, if $M \rightsquigarrow M'$ in $\lambda^{\boxdot}$, then $[\![(M)^{\dagger}]\!] = [\![(M')^{\dagger}]\!]$ in the GoI interpretation.

Corollary

For a well-typed closed term M in $\lambda^{\Box}$, if $M \rightsquigarrow M'$ in $\lambda^{\Box}$, then $[\![((M)^{\circ})^{\dagger}]\!] = [\![((M')^{\circ})^{\dagger}]\!]$ in the GoI interpretation.

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Related work

- Linear analysis of classical modal logic S4 [Schellinx '96]
  - Gives a reduction-preserving Girard trans. from classical S4, establishing a *bi-colored linear logic* with $(!_0, ?_0)$ and $(!_1, ?_1)$
  - Uses a "linear decoration" to obtain the cut-eliminiation theorem of classical S4, through that of bi-colored linear logic

- Subexponential linear logic [Nigam et al. '09, '16] and Adjoint logic [Reed '09][Licata et al. '16, '17][Pruiksma et al. '18, '19]
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  - Based on the LNL (Linear-Non-Linear) model in [Benton '94]

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Conclusion

Summary

We have presented a linear-logical reconstruction of int. S4

- Modal linear logic $\mathrm{IMELL}^{\boxed{!}}$, λ-calc $\lambda^{\boxed{!}}$, and GoI semantics
  - The key is the $\boxed{!}$-modality, an integration of ! and $\Box$
  - Our logic can reconstruct $\lambda^{\Box}$ for $\mathrm{IS4}$ of [Davies&Pfenning '01]
  - (Properties: cut-elimination, subject reduction, SN, etc.)
- (Hilbert-style axiomatization and typed combinatory logic)

Future work

- Semantical study of modal linear logic w.r.t. truth (validity)
- Extension to other int. modal logics following [Kavvos '17], considering a categorical semantics
- Extension to subexponential linear logic or adjoint logic